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Research Article

Existence of Positive Solutions of a Singular Nonlinear Boundary Value Problem

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We are concerned with the existence of positive solutions of singular second-order boundary value problem $u''(t) + f(t, u(t)) = 0$, $t \in (0, 1)$, $u(0) = u(1) = 0$, which is not necessarily linearizable. Here, nonlinearity f is allowed to have singularities at $t = 0, 1$. The proof of our main result is based upon topological degree theory and global bifurcation techniques.

1. Introduction

Existence and multiplicity of solutions of singular problem

$$\begin{gathered} u'' + f(t, u) = 0, \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where f is allowed to have singularities at $t = 0$ and $t = 1$, have been studied by several authors, see Asakawa [1], Agarwal and O'Regan [2], O'Regan [3], Habets and Zanolin [4], Xu and Ma [5], Yang [6], and the references therein. The main tools in [1–6] are the method of lower and upper solutions, Leray-Schauder continuation theorem, and the fixed point index theory in cones. Recently, Ma [7] studied the existence of nodal solutions of the singular boundary value problem

$$u'' + ra(t)f(u) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0, \tag{1.2}$$

by applying Rabinowitz's global bifurcation theorem, where a is allowed to have singularities at $t = 0, 1$ and f is linearizable at 0 as well as at ∞. It is the purpose of this paper to study the existence of positive solutions of (1.1), which is not necessarily linearizable.

Let X be Banach space defined by

$$X = \left\{ \phi \in L^1_{\mathrm{loc}}(0,1) \mid \int_0^1 t(1-t)\left|\phi(t)\right| dt < \infty \right\}, \tag{1.3}$$

with the norm

$$\|\phi\|_X = \int_0^1 t(1-t)\left|\phi(t)\right| dt. \tag{1.4}$$

Let

$$X_+ = \{\phi \in X \mid \phi(t) \geq 0, a.e.\ t \in (0,1)\},$$
$$X_p = \left\{ \phi \in X_+ \mid \int_0^1 t(1-t)\phi(t)dt > 0 \right\}. \tag{1.5}$$

Definition 1.1. A function $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1_{loc}-Carathéodory function if it satisfies the following:

(i) for each $u \in \mathbb{R}$, $g(\cdot, u)$ is measurable;

(ii) for a.e. $t \in (0,1)$, $g(t, \cdot)$ is continuous;

(iii) for any $R > 0$, there exists $h_R \in X_p$, such that

$$\left|g(t,u)\right| \leq h_R(t), \quad \text{a.e. } t \in (0,1),\ |u| \leq R. \tag{1.6}$$

In this paper, we will prove the existence of positive solutions of (1.1) by using the global bifurcation techniques under the following assumptions.

(H1) Let $f : (0,1) \times [0,\infty) \to [0,\infty)$ be an L^1_{loc}-Carathéodory function and there exist functions $a_0(\cdot)$, $a^0(\cdot)$, $c_\infty(\cdot)$, and $c^\infty(\cdot) \in X_p$, such that

$$a_0(t)u - \xi_1(t,u) \leq f(t,u) \leq a^0(t)u + \xi_2(t,u), \tag{1.7}$$

for some L^1_{loc}-Carathéodory functions ξ_1, ξ_2 defined on $(0,1) \times [0,\infty)$ with

$$\xi_1(t,u) = \circ(a_0(t)u), \quad \xi_2(t,u) = \circ\left(a^0(t)u\right), \quad \text{as } u \longrightarrow 0, \tag{1.8}$$

uniformly for a.e. $t \in (0,1)$, and

$$c_\infty(t)u - \zeta_1(t,u) \le f(t,u) \le c^\infty(t)u + \zeta_2(t,u), \tag{1.9}$$

for some L^1_{loc}-Carathéodory functions ζ_1, ζ_2 defined on $(0,1) \times [0,\infty)$ with

$$\zeta_1(t,u) = \circ(c_\infty(t)u), \quad \zeta_2(t,u) = \circ(c^\infty(t)u), \quad \text{as } u \to \infty, \tag{1.10}$$

uniformly for a.e. $t \in (0,1)$.

(H2) $f(t,u) > 0$ for a.e. $t \in (0,1)$ and $u \in (0,\infty)$.

(H3) There exists function $c_1(\cdot) \in X_p$, such that

$$f(t,u) \ge c_1(t)u, \quad \text{a.e. } t \in (0,1), \ u \in [0,\infty). \tag{1.11}$$

Remark 1.2. If $a_0(\cdot)$, $a^0(\cdot)$, $c_\infty(\cdot)$, and $c^\infty(\cdot) \in C([0,1],(0,\infty))$, then (1.8) implies that

$$\xi_1(t,u) = \circ(u), \quad \xi_2(t,u) = \circ(u), \quad \text{as } u \to 0, \tag{1.12}$$

and (1.10) implies that

$$\zeta_1(t,u) = \circ(u), \quad \zeta_2(t,u) = \circ(u), \quad \text{as } u \to \infty. \tag{1.13}$$

The main tool we will use is the following global bifurcation theorem for problem which is not necessarily linearizable.

Theorem A (Rabinowitz, [8]). *Let V be a real reflexive Banach space. Let $F : \mathbb{R} \times V \to V$ be completely continuous, such that $F(\lambda, 0) = 0$, for all $\lambda \in \mathbb{R}$. Let $a, b \in \mathbb{R}$ $(a < b)$, such that $u = 0$ is an isolated solution of the following equation:*

$$u - F(\lambda, u) = 0, \quad u \in V, \tag{1.14}$$

for $\lambda = a$ and $\lambda = b$, where $(a,0)$, $(b,0)$ are not bifurcation points of (1.14). Furthermore, assume that

$$d(I - F(a,\cdot), B_r(0), 0) \ne d(I - F(b,\cdot), B_r(0), 0), \tag{1.15}$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$\mathcal{S} = \overline{\{(\lambda,u) : (\lambda,u) \text{ is a solution of (1.14) with } u \ne 0\}} \cup ([a,b] \times \{0\}), \tag{1.16}$$

then there exists a continuum (i.e., a closed connected set) $\mathcal{C}$ of $\mathcal{S}$ containing $[a,b]\times\{0\}$, and either

(i) *$\mathcal{C}$ is unbounded in $V\times\mathbb{R}$, or*

(ii) $\mathcal{C}\cap[(\mathbb{R}\setminus[a,b])\times\{0\}]\neq\emptyset$.

To state our main results, we need the following.

Lemma 1.3 (see [1, Proposition 4.7]). *Let $a\in X_p$, then the eigenvalue problem*

$$u''+\lambda a(t)u=0,\quad t\in(0,1),$$
$$u(0)=u(1)=0 \tag{1.17}$$

has a sequence of eigenvalues as follows:

$$0<\lambda_1(a)<\lambda_2(a)<\cdots<\lambda_k(a)<\lambda_{k+1}(a)<\cdots,\quad \lim_{k\to\infty}\lambda_k(a)=\infty. \tag{1.18}$$

Moreover, for each $k\in\mathbb{N}$, $\lambda_k(a)$ is simple and its eigenfunction $\varphi_k\in C^1[0,1]$ has exactly $k-1$ zeros in $(0,1)$.

Remark 1.4. Note that $\varphi_k\in C^1[0,1]$ and $\varphi_k(0)=\varphi_k(1)=0$ for each $k\in\mathbb{N}$. Therefore, there exist constants $M_k>0$, such that

$$|\varphi_k(t)|\leq M_k t(1-t),\quad t\in[0,1]. \tag{1.19}$$

Our main result is the following.

Theorem 1.5. *Let (H1)–(H3) hold. Assume that either*

$$\lambda_1(c_\infty)<1<\lambda_1\left(a^0\right) \tag{1.20}$$

or

$$\lambda_1(a_0)<1<\lambda_1(c^\infty), \tag{1.21}$$

then (1.1) has at least one positive solution.

Remark 1.6. For other references related to this topic, see [9–14] and the references therein.

2. Preliminary Results

Lemma 2.1 (see [15, Proposition 4.1]). *For any $h\in X$, the linear problem*

$$u''(t)+h(t)=0,\quad t\in(0,1),$$
$$u(0)=u(1)=0 \tag{2.1}$$

has a unique solution $u \in W^{1,1}(0,1)$ *and* $u' \in AC_{\mathrm{loc}}(0,1)$, *such that*

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Furthermore, if $h \in X_+$, *then*

$$u(t) \ge 0, \quad t \in [0,1]. \tag{2.4}$$

Let $Y = C[0,1]$ be the Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$, and

$$E = \{u \in C[0,1] \mid u(0) = u(1) = 0\}. \tag{2.5}$$

Let $L : D(L) \subset Y \to X$ be an operator defined by

$$Lu = -u'', \quad u \in D(L), \tag{2.6}$$

where

$$D(L) = \left\{u \in W^{1,1}(0,1) \mid u'' \in X,\, u(0) = u(1) = 0\right\}. \tag{2.7}$$

Then, from Lemma 2.1, $L^{-1} : X \to C[0,1]$ is well defined.

Lemma 2.2. *Let* $a \in X_p$ *and* φ_1 *be the first eigenfunction of* (1.17). *Then for all* $u \in D(L)$, *one has*

$$\int_0^1 u''(t)\varphi_1(t)dt = \int_0^1 u(t)\varphi_1''(t)dt. \tag{2.8}$$

Proof. For any $\delta \in (0,1/2)$, integrating by parts, we have

$$\int_\delta^{1-\delta} u''(t)\varphi_1(t)dt = u'\varphi_1\Big|_\delta^{1-\delta} - u\varphi_1'\Big|_\delta^{1-\delta} + \int_\delta^{1-\delta} u(t)\varphi_1''(t)dt. \tag{2.9}$$

Since $u \in D(L)$ and $\varphi_1 \in C^1[0,1]$, then

$$\lim_{\delta\to 0} u(\delta)\varphi_1'(\delta) = \lim_{\delta\to 0} u(1-\delta)\varphi_1'(1-\delta) = 0. \tag{2.10}$$

Therefore, we only need to prove that

$$\lim_{\delta\to 0} u'(\delta)\varphi_1(\delta) = 0, \qquad \lim_{\delta\to 0} u'(1-\delta)\varphi_1(1-\delta) = 0. \tag{2.11}$$

Let us deal with the first equality, the second one can be treated by the same way. Note that $u \in D(L)$, then

$$\left(tu'(t)\right)' = u' + tu'' \in L^1(0,\delta), \tag{2.12}$$

which implies that $tu'(t) \in AC[0,\delta]$. Then $tu'(t)$ is bounded on $[0,\delta]$. Now, we claim that

$$\lim_{t\to 0} t\left|u'(t)\right| = 0. \tag{2.13}$$

Suppose on the contrary that $\lim_{t\to 0} t|u'(t)| = a > 0$, then for δ small enough, we have

$$t\left|u'(t)\right| \geq \frac{a}{2}, \quad t \in [0,\delta]. \tag{2.14}$$

Therefore,

$$\infty > \int_0^\delta \left|u'(t)\right| dt \geq \int_0^\delta \frac{a}{2t} dt = \infty, \tag{2.15}$$

which is a contradiction. Combining (1.19) with (2.13), we have

$$\left|u'(\delta)\varphi_1(\delta)\right| \leq M_1(1-\delta)\delta\left|u'(\delta)\right| \longrightarrow 0, \quad \delta \to 0. \tag{2.16}$$

This completes the proof. □

Remark 2.3. Under the conditions of Lemma 2.2, for the later convenience, (2.8) is equivalent to

$$\langle Lu, \varphi_1\rangle = \langle u, L\varphi_1\rangle. \tag{2.17}$$

Lemma 2.4 (see [1, Lemma 2.3]). *For every $\rho \in X_+$, the subset K defined by*

$$K = L^{-1}\left(\left\{\phi \in X \mid \left|\phi(t)\right| \leq \rho(t),\ a.e.\ t \in (0,1)\right\}\right) \tag{2.18}$$

is precompact in $C[0,1]$.

Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of the problem

$$Lu = \lambda f(t,u). \tag{2.19}$$

We extend the function f to an L^1_{loc}-Carathéodory function $\overline{f}$ defined on $(0,1)\times\mathbb{R}$ by

$$\overline{f}(t,u)=\begin{cases} f(t,u), & (t,u)\in(0,1)\times[0,\infty),\\ f(t,0), & (t,u)\in(0,1)\times(-\infty,0).\end{cases} \tag{2.20}$$

Then $\overline{f}(t,u)\geq 0$ for $u\in\mathbb{R}$ and a.e. $t\in(0,1)$. For $\lambda\geq 0$, let u be an arbitrary solution of the problem

$$Lu=\lambda\overline{f}(t,u). \tag{2.21}$$

Since $\lambda\overline{f}(t,u(t))\geq 0$ for a.e. $t\in(0,1)$, Lemma 2.2 yields $u(t)\geq 0$ for $t\in[0,1]$. Thus, u is a nonnegative solution of (2.19), and the closure of the set of nontrivial solutions (λ,u) of (2.21) in $\mathbb{R}^+\times E$ is exactly Σ.

Let $g:(0,1)\times\mathbb{R}\to\mathbb{R}$ be an L^1_{loc}-Carathéodory function. Let $\widehat{N}:E\to X$ be the Nemytskii operator associated with the function g as follows:

$$\widehat{N}(u)(t)=g(t,u(t)),\quad u\in E. \tag{2.22}$$

Lemma 2.5. *Let $g(t,u)\geq 0$ on $[0,1]\times\mathbb{R}$. Let $u\in D(L)$ be such that $Lu\geq\lambda\widehat{N}(u)$ in $(0,1)$, $\lambda\geq 0$. Then,*

$$u(t)\geq 0,\quad t\in(0,1). \tag{2.23}$$

Moreover, $u(t)>0, t\in(0,1)$, whenever $u\not\equiv 0$.

Let $N:E\to X$ be the Nemytskii operator associated with the function $\overline{f}$ as follows:

$$N(u)(t)=\overline{f}(t,u),\quad u\in E. \tag{2.24}$$

Then (2.21), with $\lambda\geq 0$, is equivalent to the operator equation

$$u=\lambda L^{-1}N(u),\quad u\in E, \tag{2.25}$$

that is,

$$u(t)=\lambda\int_0^1 G(t,s)N(u(s))ds,\quad u\in E. \tag{2.26}$$

Lemma 2.6. *Let (H1) and (H2) hold. Then the operator $L^{-1}N: C[0,1]\to C[0,1]$ is completely continuous.*

Proof. From (1.10) in (H1), there exists $R > 0$, such that, for a.e. $t \in (0,1)$ and $|u| > R$,

$$|\zeta_1(t,u)| \leq \frac{1}{2}c_\infty(t)u, \qquad |\zeta_2(t,u)| \leq \frac{1}{2}c^\infty(t)u. \tag{2.27}$$

Since $\overline{f}$ is an L^1_{loc}-Carathéodory function, then there exists $h_R \in X_p$, such that, for a.e. $t \in (0,1)$ and $|u| \leq R$, $|\overline{f}(t,u)| \leq h_R(t)$. Therefore, for a.e. $t \in (0,1)$ and $u \in \mathbb{R}$, we have

$$\left|\overline{f}(t,u)\right| \leq \frac{3}{2}c^\infty(t)u + h_R(t). \tag{2.28}$$

For convenience, let $T = L^{-1}N$. We first show that $T : C[0,1] \to C[0,1]$ is continuous. Suppose that $u_m \to u$ in $C[0,1]$ as $m \to \infty$. Clearly, $\overline{f}(t,u_m) \to \overline{f}(t,u)$ as $m \to \infty$ for a.e. $t \in (0,1)$ and there exists $M > 0$ such that $\|u_m\| \leq M$ for every $m \in \mathbb{N}$. It is easy to see that

$$\begin{gathered} |Tu_m(t) - Tu(t)| \leq \int_0^1 s(1-s)\left|\overline{f}(s,u_m(s)) - \overline{f}(s,u(s))\right| ds, \\ \left|\overline{f}(s,u_m(s)) - \overline{f}(s,u(s))\right| \leq 3c^\infty(s)M + 2h_R(s), \quad \text{a.e. } s \in (0,1). \end{gathered} \tag{2.29}$$

By the Lebesgue dominated convergence theorem, we have that $Tu_m \to Tu$ in $C[0,1]$ as $m \to \infty$. Thus, $L^{-1}N$ is continuous.

Let D be a bounded set in $C[0,1]$. Lemma 2.4 together with (2.28) shows that $T(D)$ is precompact in $C[0,1]$. Therefore, T is completely continuous. □

In the following, we will apply the Leray-Schauder degree theory mainly to the mapping $\Phi_\lambda : E \to E$,

$$\Phi_\lambda(u) = u - \lambda L^{-1}N(u). \tag{2.30}$$

For $R > 0$, let $B_R = \{u \in E : \|u\| < R\}$, let $\deg(\Phi_\lambda, B_R, 0)$ denote the degree of Φ_λ on B_R with respect to 0.

Lemma 2.7. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(a^0), \lambda_1(a_0)] \cap \Lambda = \emptyset$, then there exists a number $\delta_1 > 0$ with the property*

$$\Phi_\lambda(u) \neq 0, \quad \forall u \in Y : 0 < \|u\| \leq \delta_1,\ \forall \lambda \in \Lambda. \tag{2.31}$$

Proof. Suppose to the contrary that there exist sequences $\{\mu_n\} \subset \Lambda$ and $\{u_n\}$ in $Y : \mu_n \to \mu^* \in \Lambda$, $u_n \to 0$ in Y, such that $\Phi_{\mu_n}(u_n) = 0$ for all $n \in \mathbb{N}$, then, $u_n \geq 0$ in $[0,1]$.

Set $v_n = u_n/\|u_n\|$. Then $Lv_n = \mu_n\|u_n\|^{-1}N(u_n) = \mu_n\|u_n\|^{-1}f(t,u_n)$ and $\|v_n\| = 1$. Now, from condition (H1), we have the following:

$$a_0(t)u_n - \xi_1(t,u_n) \le f(t,u_n) \le a^0(t)u_n + \xi_2(t,u_n), \tag{2.32}$$

and accordingly

$$\mu_n\left(a_0(t)v_n - \frac{\xi_1(t,u_n)}{\|u_n\|}\right) \le \mu_n\frac{f(t,u_n)}{\|u_n\|} \le \mu_n\left(a^0(t)v_n + \frac{\xi_2(t,u_n)}{\|u_n\|}\right). \tag{2.33}$$

Let φ^0 and φ_0 denote the nonnegative eigenfunctions corresponding to $\lambda_1(a^0)$ and $\lambda_1(a_0)$, respectively, then we have from the first inequality in (2.33) that

$$\left\langle \mu_n\left(a_0(t)v_n - \frac{\xi_1(t,u_n)}{\|u_n\|}\right), \varphi_0\right\rangle \le \left\langle \mu_n\frac{f(t,u_n)}{\|u_n\|}, \varphi_0\right\rangle = \langle Lv_n, \varphi_0\rangle. \tag{2.34}$$

From Lemma 2.2, we have that

$$\langle Lv_n, \varphi_0\rangle = \langle v_n, L\varphi_0\rangle = \lambda_1(a_0)\langle v_n, a_0(t)\varphi_0\rangle. \tag{2.35}$$

Since $u_n \to 0$ in E, from (1.12), we have that

$$\frac{\xi_1(t,u_n)}{\|u_n\|} \longrightarrow 0, \quad \text{as } \|u_n\| \longrightarrow 0. \tag{2.36}$$

By the fact that $\|v_n\| = 1$, we conclude that $v_n \rightharpoonup v$ in E. Thus,

$$\langle v_n, a_0(t)\varphi_0\rangle \longrightarrow \langle v, a_0(t)\varphi_0\rangle. \tag{2.37}$$

Combining this and (2.35) and letting $n \to \infty$ in (2.34), it follows that

$$\langle \mu^* a_0(t)v, \varphi_0\rangle \le \lambda_1(a_0)\langle a_0(t)\varphi_0, v\rangle, \tag{2.38}$$

and consequently

$$\mu^* \le \lambda_1(a_0). \tag{2.39}$$

Similarly, we deduce from second inequality in (2.33) that

$$\lambda_1\left(a^0\right) \le \mu^*. \tag{2.40}$$

Thus, $\lambda_1(a^0) \le \mu^* \le \lambda_1(a_0)$. This contradicts $\mu^* \in \Lambda$. □

Corollary 2.8. *For $\lambda \in (0, \lambda_1(a^0))$ and $\delta \in (0, \delta_1)$, $\deg(\Phi_\lambda, B_\delta, 0) = 1$.*

Proof. Lemma 2.7, applied to the interval $\Lambda = [0, \lambda]$, guarantees the existence of $\delta_1 > 0$, such that for $\delta \in (0, \delta_1)$,

$$u - \tau\lambda L^{-1}N(u) \neq 0, \quad u \in E : 0 < \|u\| \leq \delta, \ \tau \in [0,1]. \tag{2.41}$$

This together with Lemma 2.6 implies that for any $\delta \in (0, \delta_1)$,

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(I, B_\delta, 0) = 1, \tag{2.42}$$

which ends the proof. □

Lemma 2.9. *Suppose $\lambda > \lambda_1(a_0)$, then there exists $\delta_2 > 0$ such that for all $u \in E$ with $0 < \|u\| \leq \delta_2$, for all $\tau \geq 0$,*

$$\Phi_\lambda(u) \neq \tau\varphi_0, \tag{2.43}$$

where φ_0 is the nonnegative eigenfunction corresponding to $\lambda_1(a_0)$.

Proof. Suppose on the contrary that there exist $\tau_n \geq 0$ and a sequence $\{u_n\}$ with $\|u_n\| > 0$ and $u_n \to 0$ in E such that $\Phi_\lambda(u_n) = \tau_n\varphi_0$ for all $n \in \mathbb{N}$. As

$$Lu_n = \lambda N(u_n) + \tau_n\lambda_1(a_0)a_0(t)\varphi_0 \tag{2.44}$$

and $\tau_n\lambda_1(a_0)a_0(t)\varphi_0 \geq 0$ in $(0,1)$, it concludes from Lemma 2.2 that

$$u_n(t) \geq 0, \quad t \in [0,1]. \tag{2.45}$$

Notice that $u_n \in D(L)$ has a unique decomposition

$$u_n = w_n + s_n\varphi_0, \tag{2.46}$$

where $s_n \in \mathbb{R}$ and $\langle w_n, a_0(t)\varphi_0\rangle = 0$. Since $u_n \geq 0$ on $[0,1]$ and $\|u_n\| > 0$, we have from (2.46) that $s_n > 0$.

Choose $\sigma > 0$, such that

$$\sigma < \frac{\lambda - \lambda_1(a_0)}{\lambda}. \tag{2.47}$$

By (H1), there exists $r_1 > 0$, such that

$$|\xi_1(t,u)| \leq \sigma a_0(t)u, \quad \text{a.e. } t \in (0,1), \ u \in [0, r_1]. \tag{2.48}$$

Therefore, for a.e. $t \in (0,1)$, $u \in [0, r_1]$,

$$f(t,u) \geq a_0(t)u - \xi_1(t,u) \geq (1-\sigma)a_0(t)u. \tag{2.49}$$

Since $\|u_n\| \to 0$, there exists $N^* > 0$, such that

$$0 \leq u_n \leq r_1, \quad \forall n \geq N^*, \tag{2.50}$$

and consequently

$$f(t,u_n) \geq (1-\sigma)a_0(t)u_n, \quad \forall n \geq N^*. \tag{2.51}$$

Applying (2.51), it follows that

$$\begin{aligned} s_n\lambda_1(a_0)\langle \varphi_0, a_0(t)\varphi_0\rangle &= \langle u_n, L\varphi_0\rangle = \langle Lu_n, \varphi_0\rangle \\ &= \lambda\langle N(u_n), \varphi_0\rangle + \tau_n\lambda_1(a_0)\langle a_0(t)\varphi_0, \varphi_0\rangle \\ &\geq \lambda\langle N(u_n), \varphi_0\rangle \geq \lambda\langle (1-\sigma)a_0(t)u_n, \varphi_0\rangle \\ &= \lambda(1-\sigma)\langle a_0(t)\varphi_0, u_n\rangle \\ &= \lambda(1-\sigma)s_n\langle a_0(t)\varphi_0, \varphi_0\rangle. \end{aligned} \tag{2.52}$$

Thus,

$$\lambda_1(a_0) \geq \lambda(1-\sigma). \tag{2.53}$$

This contradicts (2.47). □

Corollary 2.10. *For $\lambda > \lambda_1(a_0)$ and $\delta \in (0, \delta_2)$*, $\deg(\Phi_\lambda, B_\delta, 0) = 0$.

Proof. Let $0 < \delta \leq \delta_2$, where δ_2 is the number asserted in Lemma 2.9. As Φ_λ is bounded in $\overline{B}_\delta$, there exists $c > 0$ such that $\Phi_\lambda(u) \neq c\varphi_0$, for all $u \in \overline{B}_\delta$. By Lemma 2.9, one has

$$\Phi_\lambda(u) \neq \tau c\varphi_0, \quad u \in \partial B_\delta, \ \tau \in [0,1]. \tag{2.54}$$

This together with Lemma 2.6 implies that

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(\Phi_\lambda - c\varphi_0, B_\delta, 0) = 0. \tag{2.55}$$

□

Now, using Theorem A, we may prove the following.

Proposition 2.11. $[\lambda_1(a^0), \lambda_1(a_0)]$ *is a bifurcation interval from the trivial solution for* (2.30)*. There exists an unbounded component* $\mathcal{C}$ *of positive solutions of* (2.30) *which meets* $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$*. Moreover,*

$$\mathcal{C} \cap \left[\left(\mathbb{R} \setminus \left[\lambda_1\left(a^0\right), \lambda_1(a_0)\right]\right) \times \{0\}\right] = \emptyset. \tag{2.56}$$

Proof. For fixed $n \in \mathbb{N}$ with $\lambda_1(a^0) - (1/n) > 0$, let us take that $a_n = \lambda_1(a^0) - (1/n)$, $b_n = \lambda_1(a_0) + (1/n)$ and $\widehat{\delta} = \min\{\delta_1, \ \delta_2\}$. It is easy to check that, for $0 < \delta < \widehat{\delta}$, all of the conditions of Theorem A are satisfied. So there exists a connected component $\mathcal{C}_n$ of solutions of (2.30) containing $[a_n, \ b_n] \times \{0\}$, and either

(i) $\mathcal{C}_n$ is unbounded, or

(ii) $\mathcal{C}_n \cap [(\mathbb{R} \setminus [a_n, b_n]) \times \{0\}] \neq \emptyset$.

By Lemma 2.7, the case (ii) can not occur. Thus, $\mathcal{C}_n$ is unbounded bifurcated from $[a_n, b_n] \times \{0\}$ in $\mathbb{R} \times E$. Furthermore, we have from Lemma 2.7 that for any closed interval $I \subset [a_n, b_n] \setminus [\lambda_1(a^0), \lambda_1(a_0)]$, if $u \in \{y \in E \ \mid (\lambda, y) \in \Sigma, \lambda \in I\}$, then $\|u\| \to 0$ in E is impossible. So $\mathcal{C}_n$ must be bifurcated from $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ in $\mathbb{R} \times E$. □

3. Proof of the Main Results

Proof of Theorem 1.5. It is clear that any solution of (2.30) of the form $(1, u)$ yields solutions u of (1.1). We will show that $\mathcal{C}$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. To do this, it is enough to show that $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$. Let $(\eta_n, y_n) \in \mathcal{C}$ satisfy

$$\eta_n + \|y_n\| \longrightarrow \infty. \tag{3.1}$$

We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (2.30) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times E) = \emptyset$.

Case 1. consider the following:

$$\lambda_1(c_\infty) < 1 < \lambda_1\left(a^0\right). \tag{3.2}$$

In this case, we show that the interval

$$\left(\lambda_1(c_\infty), \lambda_1\left(a^0\right)\right) \subseteq \{\lambda \in \mathbb{R} \mid (\lambda, u) \in \mathcal{C}\}. \tag{3.3}$$

We divide the proof into two steps.

Step 1. We show that $\{\eta_n\}$ is bounded.

Since $(\eta_n, y_n) \in \mathcal{C}$, $Ly_n = \eta_n f(t, y_n)$. From (H3), we have

$$Ly_n \geq \eta_n c_1(t) y_n. \tag{3.4}$$

Let $\overline{\varphi}$ denote the nonnegative eigenfunction corresponding to $\lambda_1(c_1)$.

From (3.4), we have

$$\langle Ly_n, \overline{\varphi}\rangle \geq \eta_n \langle c_1(t) y_n, \overline{\varphi}\rangle. \tag{3.5}$$

By Lemma 2.2, we have

$$\lambda_1(c_1)\langle y_n, c_1(t)\overline{\varphi}\rangle = \langle y_n, L\overline{\varphi}\rangle \geq \eta_n \langle c_1(t)\overline{\varphi}, y_n\rangle. \tag{3.6}$$

Thus,

$$\eta_n \leq \lambda_1(c_1). \tag{3.7}$$

Step 2. We show that $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$.

From (3.1) and (3.7), we have that $\|y_n\| \to \infty$. Notice that (2.30) is equivalent to the integral equation

$$y_n(t) = \eta_n \int_0^1 G(t,s) f(s, y_n(s)) ds, \tag{3.8}$$

which implies that

$$\begin{aligned} \eta_n \int_0^1 G(t,s)\left[c^\infty(s) y_n(s) + \zeta_2(s, y_n(s))\right] ds &\geq y_n(t) \\ &\geq \eta_n \int_0^1 G(t,s)\left[c_\infty(s) y_n(s) - \zeta_1(s, y_n(s))\right] ds. \end{aligned} \tag{3.9}$$

We divide the both sides of (3.9) by $\|y_n\|$ and set $v_n = y_n/\|y_n\|$. Since v_n is bounded in E, there exist a subsequence of $\{v_n\}$ and $v^* \in E$ with $v^* \geq 0$ and $v^* \not\equiv 0$ on $(0,1)$, such that

$$\eta_n \longrightarrow \eta^*, \quad v_n \stackrel{\omega}{\rightharpoonup} v^* \quad \text{in } E, \tag{3.10}$$

relabeling if necessary. Thus, (3.9) yields that

$$\eta^* \int_0^1 G(t,s) c^\infty(s) v^*(s) ds \geq v^*(t) \geq \eta^* \int_0^1 G(t,s) c_\infty(s) v^*(s) ds. \tag{3.11}$$

Let φ^{∞} and φ_{∞} denote the nonnegative eigenfunctions corresponding to $\lambda_1(c^{\infty})$ and $\lambda_1(c_{\infty})$, respectively, then it follows from the second inequality in (3.11) that

$$\begin{aligned}
\lambda_1(c_{\infty})\langle c_{\infty}\varphi_{\infty}, v^*\rangle &= \langle L\varphi_{\infty}, v^*\rangle = \langle -\varphi_{\infty}'', v^*\rangle = -\int_0^1 \varphi_{\infty}''(t)v^*(t)dt \\
&\geq -\int_0^1 \varphi_{\infty}''(t)\eta^*\int_0^1 G(t,s)c_{\infty}(s)v^*(s)dsdt \\
&= -\eta^*\int_0^1 c_{\infty}(s)v^*(s)\int_0^1 G(t,s)\varphi_{\infty}''(t)dtds \\
&= \eta^*\int_0^1 c_{\infty}(s)v^*(s)\varphi_{\infty}(s)ds \\
&= \eta^*\langle c_{\infty}\varphi_{\infty}, v^*\rangle,
\end{aligned} \tag{3.12}$$

and consequently

$$\eta^* \leq \lambda_1(c_{\infty}). \tag{3.13}$$

Similarly, we deduce from the first inequality in (3.11) that

$$\lambda_1(c^{\infty}) \leq \eta^*. \tag{3.14}$$

Thus,

$$\lambda_1(c^{\infty}) \leq \eta^* \leq \lambda_1(c_{\infty}). \tag{3.15}$$

So $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^{\infty}), \lambda_1(c_{\infty})] \times \{\infty\}$.

Case 2. $\lambda_1(a_0) < 1 < \lambda_1(c^{\infty})$.

In this case, if $(\eta_n, y_n) \in \mathcal{C}$ is such that

$$\begin{aligned}
\lim_{n\to\infty}(\eta_n + \|y_n\|) &= \infty, \\
\lim_{n\to\infty}\eta_n &= \infty,
\end{aligned} \tag{3.16}$$

then

$$(\lambda_1(a_0), \lambda_1(c^{\infty})) \subseteq \{\lambda \in (0,\infty) \mid (\lambda, u) \in \mathcal{C}\}, \tag{3.17}$$

and moreover,

$$(\{1\} \times E) \cap \mathcal{C} \neq \emptyset. \tag{3.18}$$

Assume that $\{\eta_n\}$ is bounded, applying a similar argument to that used in Step 2 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$\eta_n \longrightarrow \eta^* \in [\lambda_1(c^\infty), \lambda_1(c_\infty)], \quad \|y_n\| \longrightarrow \infty, \quad \text{as} \quad n \longrightarrow \infty. \tag{3.19}$$

Again $\mathcal{C}$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(c^\infty), \lambda_1(c_\infty)] \times \{\infty\}$ and the result follows.

□

Remark 3.1. Lomtatidze [13, Theorem 1.1] proved the existence of solutions of singular two-point boundary value problems as follows:

$$\begin{gathered} u''(t) = g(t, u), \\ u(a) = 0, \qquad u(b) = 0, \end{gathered} \tag{3.20}$$

under the following assumptions:

(A1)

$$\begin{gathered} g(t, x) \le h_1(t)x, \quad 0 < x < \delta, \\ g(t, x) \ge h_2(t)x, \quad x > \frac{1}{\delta}, \end{gathered} \tag{3.21}$$

where $h_i : (a, b) \to R (i = 1, 2)$ satisfies the following condition:

$$\int_a^b (t-a)(b-t)|h_i(t)|dt < +\infty \quad (i = 1, 2), \tag{3.22}$$

(A2) For $i = 1, 2$, let v_i be the solution of singular IVPs

$$v''(t) = h_i(t)v, \qquad v(a) = 0, \qquad v'(a) = 1, \tag{3.23}$$

satisfying v_1 has at least one zero in $(a, b]$ and v_2 has no zeros in $(a, b]$.

It is worth remarking that (A1)-(A2) imply Condition (1.21) in Theorem 1.5. However, Condition (1.21) is easier to be verified than (A1)-(A2) since $\lambda_1(c^\infty)$ and $\lambda_1(a_0)$ are easily estimated by Rayleigh's Quotient.

The language of eigenvalue of singular linear eigenvalue problem did not occur until Asakawa [1] in 2001. The first part of Theorem 1.5 is new.

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