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ORIGINAL ARTICLE

An improved error bound on the boundary inversion for a sideways heat equation

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Abstract
We consider a new a-priori constraint for a sideways heat equation. This new constraint is used as a source condition for regularization resulting in a new computational method. An error estimate on this new regularization technique is obtained, showing an improvement over the existing methods by the traditional a-priori constraint at the boundary reconstruction.

Keywords Ill-posedness · Regularization · Error estimate · Source condition

1 Introduction

1.1 The mathematical model

In practical steel production industry, it is sometimes necessary to estimate the surface temperature or heat flux on a body from a measured temperature history at a fixed location inside the body. This is called inverse heat conduction problem (IHCP) or sideways heat problem [1].

In a one-dimensional setting, assuming that the body is large enough and the measured temperature is given at the interior location at $x = L$, this situation can be modeled as the following sideways heat problem:

$$\begin{aligned} &u_t - u_{xx} = 0, \quad x > 0, t > 0, \\ &u(x, 0) = 0, \quad x > 0, \\ &u(L, t) = g(t), \quad t > 0, \\ &u(x, t)|_{x\to\infty} \quad \text{bounded}. \end{aligned} \tag{1.1}$$

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The inverse problem under consideration is to seek the boundary solution $u(0,\cdot) \in L^2(0,\infty)$ (or $u_x(0,\cdot) \in L^2(0,\infty)$) from the given data $g(\cdot) \in L^2(0,\infty)$. We call this procedure boundary inversion.

In real application, $g(\cdot)$ contains measurement errors which results in a measurement data function $g_\delta(\cdot) \in L^2(0,\infty)$ satisfying an error tolerance level δ by

$$\|g_\delta(\cdot) - g(\cdot)\| \le \delta, \tag{1.2}$$

where the constant $\delta > 0$ represents a bound on the measurement error, $\|\cdot\|$ denotes the L^2-norm. Assume that there exists a constant $M > 0$, such that the following traditional a-priori bound exists for the above sideways problem:

$$\|u(0,\cdot)\|_p \le M, \quad p \ge 0, \tag{1.3}$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R})$ [when $p = 0$, we define $M = \tilde{M}$, see (1.8)]. This condition is also known as the a-priori condition (or the source condition) for the exact solution.

Throughout this paper, we extend all the functions to the whole line $-\infty < t < \infty$ by setting the functions to be zero for $t < 0$ if necessary. Let

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\xi t} dt \tag{1.4}$$

be the Fourier transform of the function $h(t) \in L^2(\mathbb{R})$. The solution of (1.1) can be formulated in the frequency domain:

$$\hat{u}(x,\xi) = e^{(L-x)(i\xi)^{\frac{1}{2}}} \hat{g}(\xi), \tag{1.5}$$

where

$$\eta := (i\xi)^{\frac{1}{2}} = |\xi|^{\frac{1}{2}} \left(\cos\left(\frac{\pi}{4}\right) + i \,\mathrm{sign}(\xi) \sin\left(\frac{\pi}{4}\right) \right). \tag{1.6}$$

Denote the real part a and imaginary part b of η as

$$a := \mathfrak{Re}(\eta) = \sqrt{|\xi|/2};\ b := \mathfrak{Im}(\eta) = \mathrm{sign}(\xi)\sqrt{|\xi|/2}. \tag{1.7}$$

The sideways model for heat equation has been investigated by many authors. It is well known that the sideways heat problem is an exponentially ill-posed problem [2]. All kinds of the regularization strategies were proposed to obtain a stable numerical solution for the problem. These include the Tikhonov regularization method [3], difference approximation method [4], wavelet method [5], Fourier cut-off method [6], hyperbolic approximation method [7,8], optimal filtering method [9], mollification methods [10–12], and optimal stable approximation methods [13]. The reader can refer to http://www.mai.liu.se/~frber/ip/index.html for more details.

For most of these regularization methods mentioned above, the following error estimate can be established under the a-priori condition (1.3) (e.g., see [14]):

$$\|u_\alpha^\delta(x,\cdot) - u(x,\cdot)\| \le c\delta^{\frac{x}{L}} M^{1-\frac{x}{L}} \left(\ln\frac{M}{\delta}\right)^{-2p(1-\frac{x}{L})},$$

where $u_\alpha^\delta(x,\cdot)$ is the regularization solution with noisy data $g_\delta(\cdot)$, c is a constant which is independent of M and δ.

If $x = 0$, the above error estimate becomes

$$\|u_\alpha^\delta(0,\cdot) - u(0,\cdot)\| \le cM \left(\ln\frac{M}{\delta}\right)^{-2p}.$$

For the sideways heat problem (1.1), the best possible error bound at boundary $x = 0$ has the form of $O(M(\ln(\frac{M}{\delta}))^{-2p})$ with $p > 0$ [13]. This is a notorious logarithmic error bound for ill-posed problems during the procedure of boundary inversion. If $M = 1000$ and $p = 1$, we have the error bound $M(\ln(\frac{M}{\delta}))^{-2} \le 1$ if and only if $\delta \le 10^{-40}$. If $p = 1/2$, the requirement on δ becomes dramatically severe in the sense that if δ is fixed and small, then the error bound $M(\ln(\frac{M}{\delta}))^{-2p}$ cannot be small enough to render the error bound meaningless.

In order to overcome the slow convergence rate of this notorious logarithmic error bound, Carasso in his work for solving backward heat conduction problem introduced a concept called slow time evolution from continuation boundary (SECB) [15]. Other related works on backward heat conduction problem in time and de-blurring problems can be referred to [16,17]. In this paper, we adapt the concept of SECB to deal with the sideways heat problem.

1.2 The a-priori information (or the source condition)

An a-priori information about the unknown solution has been proven to be essential in the analysis of ill-posed problems in mathematical physics. Without the a-priori information, the convergence rate of constructed regularization method is arbitrarily slow [18–21]. In this paper, the a-priori information (1.3) is interpreted as the source condition in the framework of regularization theory [2,14].

The classical regularization method is based on prescribed bounds on the derivatives of the unknown solution [e.g. (1.3)]. However, derivatives may fail to exist in many real practical problems. For example, in image deblurring problem, derivatives of the original image may not exist [16,17]. Another example is the sideways heat problem proposed in this paper, derivatives of the unknown solution $u(0, t)$ (which is the boundary data) may not exist.

Following the usage of slow time evolution from the continuation boundary (SECB) [15,22], we introduce the SECB constraint for the problem (1.1) as follow:

For any constant $K > 0$ such that $K \ll \tilde{M}/\delta$ where $\tilde{M}$ satisfies $\|u(0,\cdot)\| \le \tilde{M}$ [i.e., (1.3) when $p = 0$], define s^* by

$$s^* = \frac{L \ln\left(\frac{\tilde{M}}{\tilde{M} - K\delta}\right)}{\ln \frac{\tilde{M}}{\delta}}. \tag{1.8}$$

The slow evolution constraint applied to the sideways problem requires that there exists a known constant K and a known fixed s with $L > s > s^*$, such that

$$\|u(s,\cdot) - u(0,\cdot)\| \le K\delta. \tag{1.9}$$

Equation (1.9) is called the SECB constraint for the sideways problem (1.1). The condition $s > s^*$ will be interpreted in the subsequent Lemma 2.1. By Parseval's equality in Fourier analysis, the a-priori information (1.9) reads:

$$\int_{\mathbb{R}} |1 - e^{-s(i\xi)^{1/2}}|^2 |\hat{u}(0,\xi)|^2 d\xi \le K^2\delta^2, \tag{1.10}$$

which will hold for small value of s.

In fact, (1.10) implies

$$\int_{\mathbb{R}} |1 - e^{-s\sqrt{|\xi|/2}}|^2 |\hat{u}(0,\xi)|^2 d\xi \le K^2\delta^2. \tag{1.11}$$

This is because from (1.7) we have

$$|1 - e^{-s(i\xi)^{1/2}}|^2 \ge |1 - e^{-s\sqrt{|\xi|/2}}|^2. \tag{1.12}$$

Based on the existing SECB for backward heat conduction problem in time, in this paper we devise a new regularization method for solving the sideways heat problem. Under the new a-priori information (1.9), the error bound (2.20) improves over the traditional logarithmic error bound.

2 Regularization and an improved error estimate

Our main aim is to derive the error estimates for the problem (1.1) under the new a-priori information (1.9).

Consider the forward problem of (1.1):

$$\begin{aligned} &u_t - u_{xx} = 0, \quad x > 0, t > 0, \\ &u(x,0) = 0, \quad x > 0, \\ &u(0,t) = f(t), \quad t > 0, \\ &u(x,t)|_{x\to\infty} \quad \text{bounded}. \end{aligned} \tag{2.1}$$

We can reformulate the solution of (2.1) as an operator equation in the frequency domain:

$$\hat{B}(s)\hat{f}(\xi) = \hat{u}(s, \xi), \tag{2.2}$$

where $\hat{B}(s) = e^{-s(i\xi)^{1/2}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a multiplication operator.

For problem (2.1), in the physical domain we denote the forward operator as $B(s) : u(0, t) \to u(s, t)$ where $B(s)f(t) = \frac{s}{2\sqrt{\pi}} \int_0^t \frac{f(\tau)}{(t-\tau)^{3/2}} \exp(-\frac{s^2}{4(t-\tau)})d\tau$ and $f(t) := u(0, t)$.

From the classical Tikhonov regularization theory, if the a-priori information is given by (1.3), the Tikhonov regularization solution is obtained as the minimizer of the following functional

$$\|B(L)u(0, \cdot) - g_\delta(\cdot)\|^2 + (\delta^2/M^2)\|u(0, \cdot)\|_p^2. \tag{2.3}$$

For the a-priori information (1.9), if two regularization parameters K and s are given, the regularized solution to the problem is then the minimizer of the following functional

$$\|B(L)u(0, \cdot) - g_\delta(\cdot)\|^2 + K^{-2}\|u(s, \cdot) - u(0, \cdot)\|^2. \tag{2.4}$$

In terms of $B(s)$ we may express the solution $f^\dagger(t)$ for the sideways problem at $x = 0$ as follows:

$$f^\dagger(\cdot) = \arg\min_{f \in L^2}\{\|B(L)f(\cdot) - g_\delta(\cdot)\|^2 + K^{-2}\|(B(s) - I)f(\cdot)\|^2\}. \tag{2.5}$$

This leads to the following Euler equation for $f^\dagger(\cdot)$:

$$[B(L)^*B(L) + K^{-2}(B(s) - I)^*(B(s) - I)]f^\dagger(\cdot) = B(L)^*g_\delta(\cdot), \tag{2.6}$$

where $B(L)^*$ is the adjoint operator of $B(L)$ in L^2-space. Since $\widehat{B(s)f} = e^{-s(i\xi)^{1/2}}\hat{f}(\xi)$, we can obtain the solution $f^\dagger(\cdot)$ in closed form in the Fourier transform domain with noisy data. Hence, we have

$$\hat{f}^\dagger(\xi) = \frac{\overline{e^{-L(i\xi)^{1/2}}}}{e^{-L\sqrt{2|\xi|}} + K^{-2}|1 - e^{-s(i\xi)^{1/2}}|^2}\hat{g}_\delta(\xi), \tag{2.7}$$

which gives $f^\dagger(t)$ by using inverse Fourier transform. Here, $\bar{v}$ denotes the complex conjugate of the complex function v.

The error estimate for the above proposed regularization method is given in the following theorem.

Theorem 2.1 *Suppose that the solution $f(t) := u(0, t)$ satisfies* (1.2) and (1.9) *and $f^\dagger(t)$ denotes the regularized solution. Let $P = P(L, K, s)$ be the positive definite self-adjoint operator in $L^2(\mathbb{R})$ given by*

$$P = B^*(L)B(L) + K^{-2}(B(s) - I)^*(B(s) - I). \tag{2.8}$$

Then, $f^\dagger$ is the unique solution of $Pf^\dagger = B(L)^* g_\delta$ satisfying

$$\|B(L)f^\dagger - g_\delta\|^2 + K^{-2}\|(B(s)-I)f^\dagger\|^2 \le 2\delta^2, \tag{2.9}$$

$$\|B(L)(f - f^\dagger)\|^2 + K^{-2}\|(B(s)-I)(f - f^\dagger)\|^2 \le 2\delta^2, \tag{2.10}$$

$$\|f^\dagger(\cdot) - f(\cdot)\| \le \sqrt{2}\delta\|P^{-\frac{1}{2}}\|. \tag{2.11}$$

Proof Let $\mathcal{H}$ denote the Hilbert space direct sum $L^2(\mathbb{R}) \bigoplus L^2(\mathbb{R})$ with elements $[u, v]$ and $([u_1, v_1], [u_2, v_2]) := \langle u_1, u_2\rangle + \langle v_1, v_2\rangle$ represent the scalar product of a pair of elements with corresponding norm $|||[u_1, v_1]|||^2 := \|u_1\|^2 + \|v_1\|^2$. Define $C : L^2(\mathbb{R}) \to \mathcal{H}$ to be $Ch = [B(L)h, K^{-1}(B(s) - I)h]$ and $\tilde{g} = [g_\delta, 0]$. Minimizing $|||Ch - \tilde{g}|||$ over all $h \in L^2(\mathbb{R})$, we get the normal equation $C^*Cf^\dagger = C^*\tilde{g}$, which gives $Pf^\dagger = B(L)^*g_\delta$. From (1.2) and (1.9), we have $|||Cf - g_\delta|||^2 \le 2\delta^2$. The minimizer $f^\dagger$ is the element such that $Cf^\dagger$ is the orthogonal projection in $\mathcal{H}$ of $\tilde{g}$. By the Pythagorean theorem, we have

$$|||Cf^\dagger - \tilde{g}|||^2 + |||C(f - f^\dagger)|||^2 = |||Cf - \tilde{g}|||^2 \le 2\delta^2.$$

Hence, $|||Cf^\dagger - \tilde{g}|||^2 \le 2\delta^2$ and $|||C(f - f^\dagger)|||^2 \le 2\delta^2$ which gives (2.9) and (2.10), respectively. Since P is a self-adjoint operator in $L^2(\mathbb{R})$, we have

$$\|P^{\frac{1}{2}}(f - f^\dagger)\|^2 = \langle P(f - f^\dagger), (f - f^\dagger)\rangle = |||C(f - f^\dagger)|||^2 \le 2\delta^2.$$

Therefore,

$$\|f - f^\dagger\| = \|P^{-\frac{1}{2}}P^{\frac{1}{2}}(f - f^\dagger)\| \le \|P^{-\frac{1}{2}}\|\|P^{1/2}(f - f^\dagger)\| \le \sqrt{2}\delta\|P^{-\frac{1}{2}}\|.$$

□

To obtain the error estimate for the regularized solution, we need to estimate the norm $\|P^{-\frac{1}{2}}\|$. Using the Parseval's identity in Fourier analysis, we have $\|P^{-\frac{1}{2}}h\| = \|\widehat{P^{-\frac{1}{2}}h}\|$. It yields that

$$\|P^{-\frac{1}{2}}\| = \sup_{\xi\in\mathbb{R}}\left[e^{-L\sqrt{2|\xi|}} + K^{-2}|1 - e^{-s\sqrt{i\xi}}|^2\right]^{-1/2}. \tag{2.12}$$

From (1.12), there holds

$$e^{-L\sqrt{2|\xi|}} + K^{-2}|1 - e^{-s\sqrt{i\xi}}|^2 \ge e^{-L\sqrt{2|\xi|}} + K^{-2}|1 - e^{-s\sqrt{|\xi|/2}}|^2,$$

we have

$$\sup_{\xi\in\mathbb{R}}\left[e^{-L\sqrt{2|\xi|}}+K^{-2}|1-e^{-s\sqrt{i\xi}}|^2\right]^{-1/2} \le \sup_{\xi\in\mathbb{R}}\left[e^{-L\sqrt{2|\xi|}}+K^{-2}|1-e^{-s\sqrt{|\xi|/2}}|^2\right]^{-1/2}. \tag{2.13}$$

Now, we need to estimate the term

$$\sup_{\xi\in\mathbb{R}}\left[e^{-L\sqrt{2|\xi|}}+K^{-2}|1-e^{-s\sqrt{|\xi|/2}}|^2\right]^{-1/2}$$
$$=\left[\inf_{x\geq 0}(e^{-2Lx}+K^{-2}(1-e^{-sx})^2)\right]^{-1/2}. \qquad (2.14)$$

Finally, for $|\xi|$ sufficiently large, we have

$$\|P^{-\frac{1}{2}}\|\approx\left[\inf_{x\geq 0}(e^{-2Lx}+K^{-2}(1-e^{-sx})^2)\right]^{-1/2}:=\Phi(x). \qquad (2.15)$$

The key point of the next step is to minimize the function $\upsilon(x)$ with $x\geq 0$, where

$$\upsilon(x)=e^{-2Lx}+K^{-2}(1-e^{-sx})^2.$$

The following lemma shows the reason why we choose $s>s^*$ in the SECB constraint (1.9).

Lemma 2.1 *Let s^* be given as in (1.8). If $s\leq s^*$, then*

$$\Phi(x)\geq\tilde{M}/(\sqrt{2}\delta),$$

where $\Phi(x)$ is defined in (2.15).

Proof Let $x_0=\frac{1}{L}\ln(\frac{\tilde{M}}{\delta})$. Then $e^{-2Lx_0}=\delta^2/\tilde{M}^2$ and $1-e^{-sx_0}\leq 1-e^{-s^*x_0}=K\delta/\tilde{M}$. Also, $\delta^2/\tilde{M}^2\leq\upsilon(x_0)\leq 2\delta^2/\tilde{M}^2$ if $s\leq s^*$. The result is now trivial from (2.15). □

Remark 2.1 If $s\leq s^*$, from (2.11) the regularization method has no convergence result for $\delta\to 0$. In Lemma 2.1, we note that for $|\xi|$ large enough, we have $e^{-L\sqrt{2|\xi|}}+K^{-2}|1-e^{-s\sqrt{i\xi}}|^2\approx e^{-L\sqrt{2|\xi|}}+K^{-2}|1-e^{-s\sqrt{|\xi|/2}}|^2$.

Lemma 2.2 *Let $0<s<L$. For $x>0$, $\upsilon(x)$ has a unique minimum at $x=\tilde{x}>0$, where $\tilde{x}$ satisfies*

$$e^{-2L\tilde{x}}=sK^{-2}(e^{-s\tilde{x}}-e^{-2s\tilde{x}}). \qquad (2.16)$$

Proof Noting the function $\upsilon(x)$ and by simple calculation for $\upsilon'(x)=0$, we can show that $\tilde{x}$ satisfies

$$e^{-2L\tilde{x}}=sK^{-2}(e^{-s\tilde{x}}-e^{-2s\tilde{x}}). \qquad \text{(a)}$$

Similarly, we can compute the second derivative

$$\upsilon''(\tilde{x})=4L^2e^{-2L\tilde{x}}-2s^2K^{-2}e^{-s\tilde{x}}+4s^2K^{-2}e^{-2s\tilde{x}}.$$

Due to (a), we have $2s^2K^{-2}e^{-s\tilde{x}} = 2sLe^{-2L\tilde{x}} + 2s^2K^{-2}e^{-2s\tilde{x}}$, then $\upsilon''(\tilde{x}) = 2L(2L-s)e^{-2L\tilde{x}} + 2s^2K^{-2}e^{-2s\tilde{x}}$. Since $0 < s < L$, it yields $\upsilon''(\tilde{x}) > 0$. Hence, $\tilde{x}$ is unique. □

Lemma 2.3 *If $K/s > 1$, then $\tilde{x}$ satisfies*

$$\frac{2e}{1+2Le}\ln\frac{K}{s} \le \tilde{x} \le \frac{1}{2L-s}\ln\frac{K^2}{s(1-(s/K)^{(2es)/(1+2Le)})}, \tag{2.17}$$

where e is the Euler's number.

Proof We take two steps to prove the inequalities.

(I) The lower bound of $\tilde{x}$.

Since the unique minimum $\tilde{x}$ of the function $\upsilon(x)$ satisfies $e^{-2L\tilde{x}} = [s^2K^{-2}\tilde{x}(e^{-s\tilde{x}} - e^{-2s\tilde{x}})]/(s\tilde{x})$ [see (2.16)] and $\theta(y) := (e^{-y} - e^{-2y})/y$ is a monotone decreasing function with the maximum value of 1 at $y = 0$, we obtain $e^{-2L\tilde{x}} \le s^2K^{-2}\tilde{x}$. Hence,

$$\tilde{x}e^{2L\tilde{x}} \ge K^2/s^2. \quad \text{(b)}$$

Taking the operator of logarithm on the two sides of (b), we get $\tilde{x}(L + \frac{\ln\tilde{x}}{2\tilde{x}}) \ge \ln\frac{K}{s}$. In order to get the lower bound of $\tilde{x}$, let $\varsigma(y) := L + \frac{\ln y}{2y}$ with $y > 0$. Then, $\varsigma(y)$ attains its maximum value at $y = e$, i.e., $\varsigma(y) \le L + 1/(2e) = \frac{1+2eL}{2e}$. Thus, $\tilde{x}\frac{1+2eL}{2e} \ge \tilde{x}(L + \frac{\ln\tilde{x}}{2\tilde{x}}) \ge \ln\frac{K}{s}$, i.e., $\tilde{x} \ge \frac{2e}{1+2eL}\ln\frac{K}{s}$. Thus we get the lower bound of $\tilde{x}$.

(II) The upper bound of $\tilde{x}$.

To obtain an upper bound of $\tilde{x}$, we first have $1 - e^{-s\tilde{x}} \ge 1 - e^{-s\frac{2e}{1+2Le}\ln\frac{K}{s}} = 1 - (s/K)^{(2es)/(1+2Le)}$. From (2.16), $e^{-(2L-s)\tilde{x}} = sK^{-2}(1 - e^{-s\tilde{x}})$, we have $sK^{-2}(1 - e^{-s\tilde{x}}) \ge sK^{-2}[1 - (s/K)^{(2es)/(1+2Le)}]$, i.e., $e^{-(2L-s)\tilde{x}} \ge sK^{-2}[1 - (s/K)^{(2es)/(1+2Le)}]$. Taking the operator of logarithm, we can get the result. □

Lemma 2.4 *If $0 < K/s \le 1$, then $\|P^{-\frac{1}{2}}\| \le L + 1$.*

Proof From (2.16), $\tilde{x}e^{2L\tilde{x}} = (K^2s^{-2})[(s\tilde{x})/(e^{-s\tilde{x}} - e^{-2s\tilde{x}})] \le (s\tilde{x})/(e^{-s\tilde{x}} - e^{-2s\tilde{x}})$ if $K/s < 1$. Hence,

$$\tilde{x}e^{(2L-s)\tilde{x}} \le (s\tilde{x})/(1 - e^{-s\tilde{x}}).$$

In order to investigate an upper bound of $\tilde{x}$, noting that $\ell(y) := y/(1 - e^{-y})$ is a monotone increasing function for $y \ge 0$ and $0 < s < L$, we get $\tilde{x}e^{(2L-s)\tilde{x}} \le (L\tilde{x})/(1 - e^{-L\tilde{x}})$. Therefore, $e^{L\tilde{x}} \le e^{(2L-s)\tilde{x}} \le L/(1 - e^{-L\tilde{x}})$, i.e., $\tilde{x} \le \frac{1}{L}\ln(L+1)$.

Thus, $e^{-2L\tilde{x}} \ge (L+1)^{-2}$. From (2.15), we have the conclusion. □

Theorem 2.2 *Suppose that the solution $f(t) := u(0,t)$ satisfies* (1.2) and (1.9) *and $f^\dagger(t)$ is the regularization solution. If $s > s^*$ and $0 < K/s < 1$, then $\|f - f^\dagger\| \le \sqrt{2}(L+1)\delta$. If $s > s^*$ and $K/s > 1$, denote two constants*

$$\mathbf{N} := [sK^{-2}(1 - (s/K)^{\frac{2es}{1+2Le}})]^{\frac{2L}{2L-s}}, \tag{2.18}$$

$$\mathbf{Q} := K^{-2}[1 - (s/K)^{\frac{2es}{1+2Le}}]^2. \tag{2.19}$$

Then, we have

$$\|f^\dagger(\cdot) - f(\cdot)\| \leq \sqrt{2}\delta(\mathbf{N} + \mathbf{Q})^{-\frac{1}{2}}. \tag{2.20}$$

Proof If $0 < K/s \leq 1$, the result is obvious from Lemma 2.4. If $K/s > 1$, we need to estimate the value of $\upsilon(\tilde{x})$. Using the upper and lower bounds for $\tilde{x}$ in Lemma 2.3, we get $\upsilon(\tilde{x}) \geq e^{-2L\frac{1}{2L-s}\ln\frac{K^2}{s(1-(s/K)^{(2es)/(1+2Le)})}} + K^{-2}(1 - e^{-s\frac{2e}{1+2Le}\ln\frac{K}{s}})^2 = \mathbf{N} + \mathbf{Q}$. The result now follows from Theorem 2.1 and the fact that $\|P^{-\frac{1}{2}}\| \leq [\upsilon(\tilde{x})]^{-\frac{1}{2}}$. □

Remark 2.2 In the proof of Theorem 2.2, an emphasis has to be made: the constant $s^* \in (0, L)$, which depends on the a-priori bound $\tilde{M}$ as well as δ and K, is not required. The knowledge of an a-priori bound $\tilde{M}$ will still be useful for the computation of s^* that will guarantee the choice of $s > s^*$, which is a premise of Theorem 2.2.

Remark 2.3 Obviously, the error estimate (2.20) improves over the classical error estimate under the assumption (1.3): $\|u_\alpha^\delta(0,\cdot) - u(0,\cdot)\| \leq cM(\ln\frac{M}{\delta})^{-2p}$ with $p \geq 0$, where $u_\alpha^\delta(0,\cdot)$ denotes the regularization solution with noisy data $g_\delta(\cdot)$. If $p = 0$, then $M = \tilde{M}$ and the a-priori information becomes $\|u(0,\cdot)\| \leq \tilde{M}$, thus the classical error estimate is $\|u_\alpha^\delta(0,\cdot) - u(0,\cdot)\| \leq c\tilde{M}$ which is not convergent.

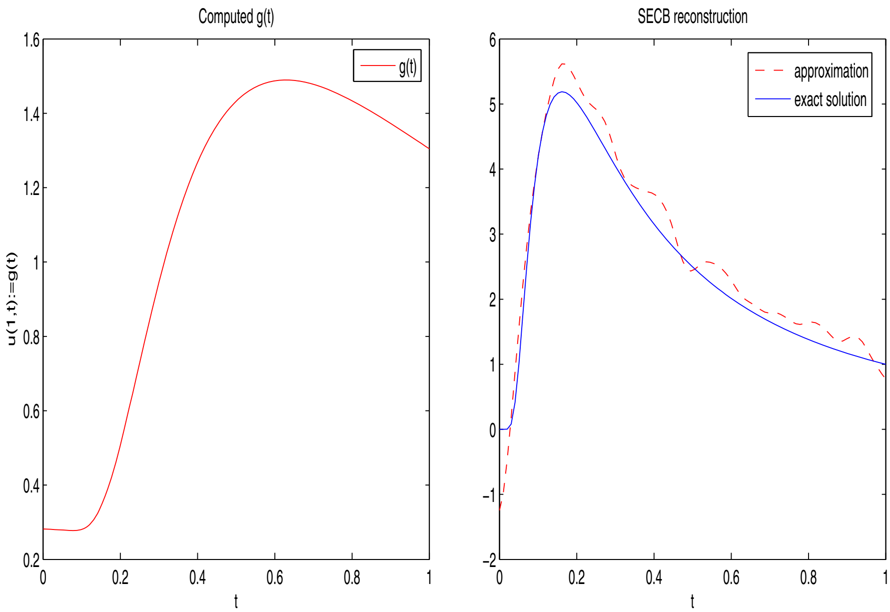
Remark 2.4 In image deblurring problem, which is also severely ill-posed, Carasso [15] observed that the SECB method sharply reduces noise contamination and works better than the classical regularization methods under the assumption (1.3).

Remark 2.5 In this paper, as an example, we used the SECB method for solving a sideways heat equation. However, following the same route, the method can also be applied to the Cauchy problem of elliptic equations and the other ill-posed problems which can be formulated in the frequency domain [23,24].

3 Numerical experiments

In this section, we present some numerical experiments to illustrate the properties of the proposed method with the fixed parameter $L = 1$.

The numerical tests are performed in the following way: First we select a solution $u(0,t) = f(t), 0 \leq t \leq 1$ and compute the data function $u(1,t) = g(t)$ by solving a well-posed quarter plane problem for the equation using a finite difference scheme. Then we add a normally distributed perturbation of variance ϵ to the data function, giving g_δ. We compute the noise level δ by discrete L^2-norm $\|g - g_\delta\|$ and the a-priori bound $\|f\| \approx \tilde{M}$ with total $m = 100$ test points . From the noisy data function we reconstruct $u(0,t)$ and compare the result with the exact solution. In the process of reconstruction, we use FFT (Fast Fourier Transform) algorithm to compute the closed form (2.7), then we use inverse FFT to obtain the reconstructed solution. On the Fourier-based method for solving sideways heat equation, the reader can consult [5].



(a) **(b)**

Fig. 1 Example 1 **a** the input data $g(t)$, **b** the reconstructed result

We conduct two tests: The first test is to show that the proposed method works well, meanwhile we give the comparison on the error bounds under the a-priori information $\|u(0,t)\| \leq \tilde{M}$: (1) the quantity $E_1 := \sqrt{2}\delta(\mathbf{N}+\mathbf{Q})^{-\frac{1}{2}}$ is the error bound for our method; (2) the quantity $E_2 := 2\tilde{M}$ is the error bound for the classical methods (e.g., see [14]). The second is to show that the theoretical requirements $s > s^*$ and (1.9) are important. As Remark 2.1 mentioned, if $s > s^*$ is violated, then we has no convergent result. We will see that the numerical solutions deteriorate as the requirements are violated gradually.

Test 1 In this test, we select three different exact solutions.

Example 1 First we select the exact solution defined on [0, 1] which is infinitely smooth:

$$f(t) = \begin{cases} 0, & \text{if} \quad t = 0, \\ \frac{1}{t^{\frac{3}{2}}} \exp\left(-\frac{(1-t)^2}{4t}\right), & \text{else}. \end{cases} \tag{3.1}$$

First by solving a well-posed forward problem, we can get the input data $g(t)$ which is displayed in Figure 1a, and then we compute the regularized solution which is displayed in Figure 1b. The parameters involved are listed as follows: $\epsilon = 1 \times 10^{-2}$, $\delta = 1 \times 10^{-2}$, $\tilde{M} = 2.89$, $s = 0.1$, $s^* = 0.02$, $K = 40$.

Example 2 Second, let us consider the exact solution defined on [0, 1] which is not continuous:

$$f(t) = \begin{cases} 1, & \text{if} \quad 0.25 \leq t \leq 0.75, \\ 0, & \text{else}. \end{cases} \tag{3.2}$$

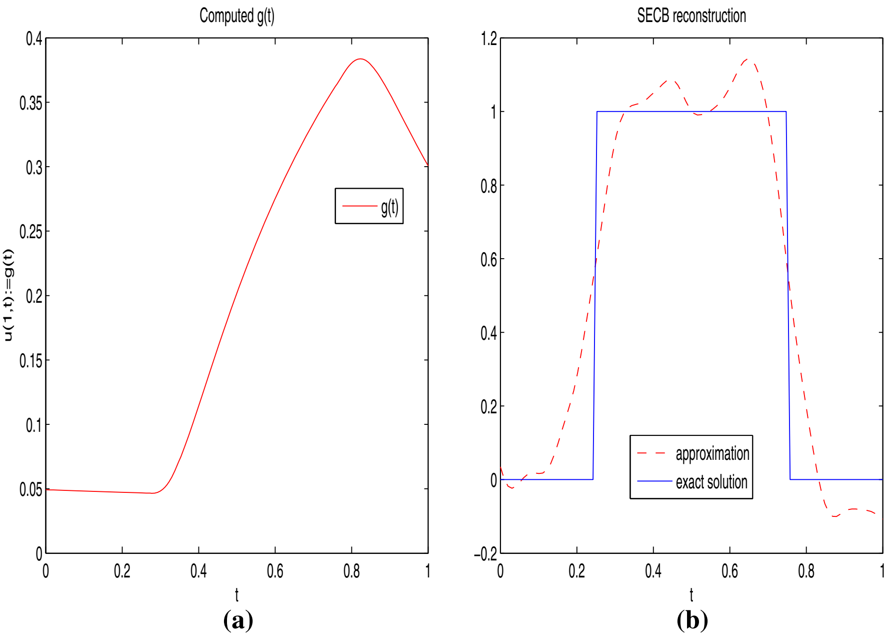


Fig. 2 Example 2 **a** the input data $g(t)$, **b** the reconstructed result

Similarly we can get the input data $g(t)$ which is displayed in Fig. 2a, and then we computed the regularized solution which is displayed in Fig. 2b. The parameters involved are listed as follows: $\epsilon = 1 * 10^{-2}$, $\delta = 1 \times 10^{-2}$, $\tilde{M} = 0.70$, $s = 0.1$, $s^* = 0.06$, $K = 15$.

Example 3 In the last example, we consider an exact solution which is not infinitely smooth

$$f(t) = \begin{cases} 4x - 1, & \text{if} \quad 0.25 \leq t < 0.50, \\ 3 - 4x, & \text{if} \quad 0.50 \leq t < 0.75, \\ 0, & \text{else}, \end{cases} \tag{3.3}$$

and the corresponding results are displayed in Figure 3a, Figure 3b. The corresponding parameters involved are listed as follows: $\epsilon = 1 * 10^{-2}$, $\delta = 1 \times 10^{-2}$, $\tilde{M} = 0.40$, $s = 0.1$, $s^* = 0.07$, $K = 10$.

For comparison, we summarize the error bounds E_1, E_2 as Table 1.

From Table 1, we see that the new error bound improves over the classical error bound.

A common characteristic of these examples is that the approximation errors are maximal at the boundary ($t = 0$ and $t = 1$) of the considered interval. A possible explanation is the extension to zero of the data function outside the time interval which introduces additional jumps into the data. We have tested our method to various other examples and got similar good numerical results.

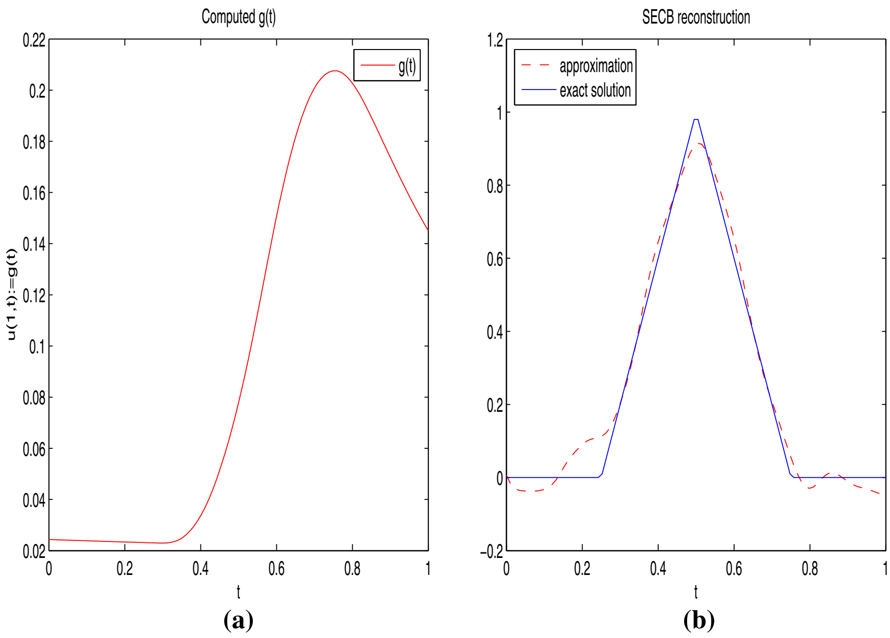


Fig. 3 Example 3 **a** the input data $g(t)$, **b** the reconstructed result

Table 1 Comparison of error bounds

Exact solution	(3.1)	(3.2)	(3.3)
E_1	1.30	0.58	0.40
E_2	5.80	1.40	0.80

Test 2 In Example 2, if we change the parameter $s = 0.01 < s^* = 0.2$ such that the requirement $s > s^*$ is violated however we keep (1.9) effective, the numerical result is displayed as Fig. 4. The parameters involved are listed as follows: $\epsilon = 1 \times 10^{-2}$, $\delta = 1 \times 10^{-2}$, $M = 0.70$, $s = 0.01$, $s^* = 0.2$, $K = 40$. From Fig. 4, we can see that the numerical result is bad, which agrees with the theoretical indication. We observed similar results for Examples 1 and 3.

All numerical tests show that the parameters s, K play important roles for the regularized solution. Usually we call s, K regularization parameters. Therefore selecting appropriate regularization parameters is an important task.

4 Conclusions

We proposed a new method for solving a sideways heat equation which is severely ill-posed. In this paper, we proved that the method is stable and gave an error bound. It is very simple and fast since the regularized solution has a closed form in the frequency domain. The numerical experiments for test examples are convincing. The method can

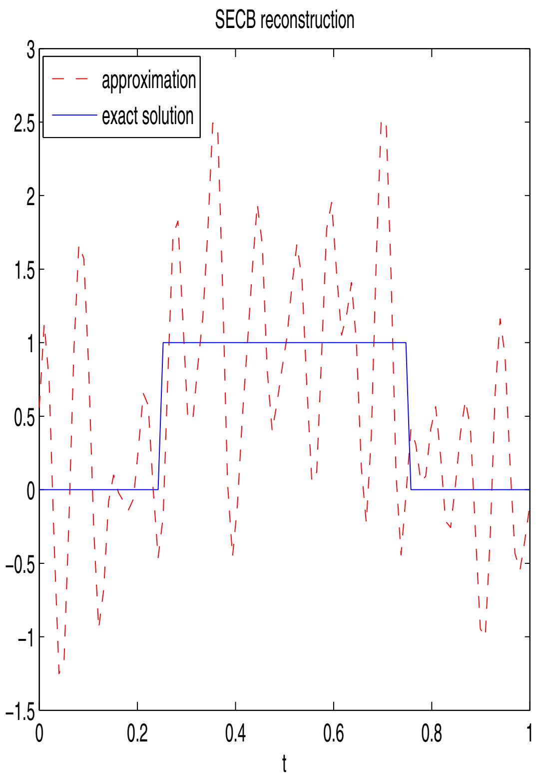


Fig. 4 The bad reconstructed result for Example 2

be generalized to multi-dimensional cases when the Fourier transform technique for the problem is applicable. However, how to choose the regularization parameters s, K accurately is an open problem. We will develop an iterative method to solve it in the forthcoming paper.

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