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Research Article

Weighted Moving Averages for a Series of Fuzzy Numbers Based on Nonadditive Measures with $\sigma - \lambda$ Rules and Choquet Integral of Fuzzy-Number-Valued Function

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The aim of this study is to generalize moving average by means of Choquet integral. First, by employing nonadditive measures with $\delta - \lambda$ rules, the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. Finally, some properties are investigated by means of convolution formula of Choquet integral. It shows that the results obtained in this paper extend the previous conclusions.

1. Introduction

The concept of nonadditive measures was originally proposed by Sugeno [1]. It replaces additivity in classical additive measures with monotonicity and can be regarded as an extension of classical additive measures. Indeed, nonadditive measures can be used to describe interdependent or interactive characteristics of information in practical applications. The Choquet integral, initiated by Choquet [2], provides a mechanism to integrate function on the basis of nonadditive measures and is a powerful technique to address interdependence and interaction among information. In fact, the Choquet integral [2] with respect to nonadditive measures has successful application in pattern recognition [3], decision-making [4–7], information fusion [8–10], economic theory [11], and so on.

Another key mathematical structure to cope with imperfect or imprecise information is a fuzzy set, developed by Zadeh [12]. Fuzzy numbers [13], a specific format of fuzzy sets, are utilized to express values in practical situation where the exact values may not be determined because of lack or imperfection of information [14]. That is, fuzzy numbers take into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty and have been used as a way of modeling uncertain and incomplete systems. Fuzzy numbers have been investigated intensively by research studies [15–17] from various aspects since it was introduced.

Motivated by the ability of Choquet integral with respect to nonadditive measures in handling interaction among information and the merit of fuzzy number in depicting uncertainty, it is of both theoretical and practical importance to combine them together and apply the combination to moving average. In this work, we want to give more insight into issues connected with the weighted moving averages for a series of fuzzy numbers based on nonadditive measures with $\sigma - \lambda$ rules by the new tools, Choquet integral and fuzzy number. This is a new contribution to our previous work [18], in which the moving average for a series of fuzzy numbers based on nonadditive measures with $\sigma - \lambda$ rules is proposed and discussed. The aim of this paper is to show that the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. Finally, some properties are investigated by means of the convolution formula of Choquet integral.

The structure of this paper is as follows. In Section 2, we review some basic concepts and properties about nonadditive measure with $\sigma-\lambda$ rules and fuzzy numbers. And the definition of product between a nonnegative matrix and fuzzy number vector is given to make our analysis possible. In Section 3, it shows that the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Meanwhile, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined and their properties are investigated by means of the convolution formula of Choquet integral. The paper ends with conclusion in Section 4.

2. Preliminaries

In this section, some basic notations and concepts of HFLTS and DTRS are briefly reviewed. Throughout this study, R^m denotes the m-dimension real Euclidean space and $R^+ = (0, \infty)$.

Definition 1 (see [1, 19, 20]). Let X denote a nonempty set and $\mathscr{A}$, a $\sigma-$ algebra on the X. A set function μ is referred to as a regular fuzzy measure if

(1) $\mu(\varnothing) = 0$

(2) $\mu(X) = 1$

(3) For every A and $B \in \mathscr{A}$ such that $A \subseteq B$, $\mu(A) \leq \mu(B)$

Definition 2 (see [1, 19, 20]). g_λ is called a fuzzy measure based on $\sigma-\lambda$ rules if it satisfies

$$g_\lambda\left(\bigcup_{i=1}^{\infty} A_i\right) = \begin{cases} \dfrac{1}{\lambda}\left\{\displaystyle\prod_{i=1}^{\infty}\left[1+\lambda g_\lambda(A_i)\right]-1\right\}, & \lambda \neq 0, \\ \displaystyle\sum_{i=1}^{\infty} g_\lambda(A_i), & \lambda = 0, \end{cases} \tag{1}$$

where $\lambda \in (-(1/\sup\mu), \infty) \cup \{0\}$, $\{A_i\} \subset \mathscr{A}$, and $A_i \cap A_j = \varnothing$ for all $i, j = 1, 2, \ldots$ and $i \neq j$.

Particularly, if $\lambda = 0$, then g_λ is a classic probability measure.

A regular fuzzy measure μ is called Sugeno measure based on $\sigma-\lambda$ rules if μ satisfies $\sigma-\lambda$ rules, briefly denoted as g_λ. The fuzzy measure denoted in this paper is Sugeno measure.

Remark 1. In Definition 2, if $n = 2$, then

$$\mu(A \cup B) = \begin{cases} \mu(A) + \mu(B) + \lambda\mu(A)\mu(B), & \lambda \neq 0, \\ \mu(A) + \mu(B), & \lambda = 0. \end{cases} \tag{2}$$

Remark 2. If X is a finite set, for any subset A of X, then

$$g_\lambda(A) = \begin{cases} \dfrac{1}{\lambda}\left\{\displaystyle\prod_{x\in A}\left[1+\lambda g_\lambda(\{x\})\right]-1\right\}, & \lambda \neq 0, \\ \displaystyle\sum_{i=1}^{\infty} g_\lambda(\{x\}), & \lambda = 0. \end{cases} \tag{3}$$

Remark 3 (see [19]). If X is a finite set, then the parameter λ of a regular Sugeno measure based on $\sigma-\lambda$ rules is determined by the following equation:

$$\prod_{i=1}^{n}\left(1+\lambda g_{\lambda i}\right) = 1 + \lambda. \tag{4}$$

Let g_λ be a fuzzy measure satisfying $\sigma-\lambda$ rules. Denoting $A = \{x_1, x_2, \ldots, x_m\} \in \mathscr{A}$, $f: A \longrightarrow \mathbb{R}$ be real-valued function, and then, the Choquet integral of f on A is defined as follows [1]:

$$(c)\int_A f\,\mathrm{d}g_\lambda = \sum_{i=1}^{m} f(x_i)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \tag{5}$$

where $A_i = \{x_i, x_{i+1}, \ldots, x_m\}$, $i = 1, 2, \ldots, m$, and $f(x_1) \leq f(x_2) \leq f(x_m)$, $i = 1, 2, \ldots, m$.

Let $g_\lambda(\{x_i\}) = g_i$, $i = 1, 2, \ldots, m$; then, $g_\lambda(A_i)$ is obtained from the following recurrence relation:

$$\begin{aligned} g_\lambda(A_m) = g_\lambda(\{x_m\}) = g_m, g_\lambda(A_i) = g_\lambda(A_{i+1}) \\ + \lambda g_i g_\lambda(A_{i+1}), \quad 1 \leq i < m. \end{aligned} \tag{6}$$

Let $\tilde{A}(x) \in \tilde{E}$, $r \in (0, 1]$ and $[\tilde{A}]^r = \{x \in R: u_{\tilde{A}}(X) \geq r\}$. $\tilde{A}$ satisfies the following:

(1) $\tilde{A}$ is a normal fuzzy set, i.e., an $x_0 \in R$ exists such that $u_{\tilde{A}}(x_0) = 1$

(2) $\tilde{A}$ is a convex fuzzy set, i.e., $u_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{u_{\tilde{A}}(x), u_{\tilde{A}}(y)\}$ for any $x, y \in R$ and $\lambda \in (0, 1]$

(3) $\tilde{A}$ is a upper semicontinuous fuzzy set

(4) $[A]^0 = \overline{X \in R: u_{\tilde{A}}(x) > 0} = \overline{\cup_{r\in(0,1]}[A]^r}$ is compact, where $\bar{A}$ denotes the closure of A

Then, $\tilde{A}$ is called a fuzzy number. We use $\tilde{E}$ to denote the fuzzy number space [21].

It is clear that each $x \in R$ can be considered as a fuzzy number $\tilde{A}$ defined by

$$u_{\tilde{A}}(x) = \begin{cases} 1, & x = A, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

Given any two fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$, k, $k_1 \mathscr{L} \neg k_2 \geq 0$, the operational rules are as follows:

(1) $k(\tilde{A}_1 + \tilde{A}_2) = k\tilde{A}_1 + k\tilde{A}_2$

(2) $k_1(k_2\tilde{A}_1) = (k_1k_2)\tilde{A}_1$

(3) $(k_1 + k_2)\tilde{A}_1) = k_1\tilde{A}_1 + k_2\tilde{A}_1$

Lemma 1 (see [21–23]). *For a fuzzy set $\tilde{A}$, it satisfies the following equation:*

$$\widetilde{A}=\bigcup_{r\in[0,1]}\left(r^{*}\cap[\widetilde{A}]^{r}\right), \tag{8}$$

where r^* denotes the fuzzy set whose membership function is a constant function r.

Let $\widetilde{A},\widetilde{B}\in\widetilde{E}$ and $k\in\mathbb{R}$; the addition and scalar conduct are defined by

$$\begin{aligned}[\widetilde{A}+\widetilde{B}]^{r}&=[\widetilde{A}]^{r}+[\widetilde{B}]^{r},\\ [k\widetilde{A}]^{r}&=k[\widetilde{A}]^{r},\end{aligned} \tag{9}$$

respectively, where $[\tilde{A}]^r=\{x\colon u_{\tilde{A}}(x)\geq r\}=[A^{-}(r),A^{+}(r)]$, for any $r\in(0,1]$.

Lemma 2 (see [21–23]). *If $\tilde{A}\in\tilde{E}$, then*

(1) $[\tilde{A}]^r$ is a nonempty bounded closed interval for any $r\in(0,1]$.

(2) $[\widetilde{A}]^{r_1}\supset[\widetilde{A}]^{r_2}$ where $0\leq r_1\leq r_2\leq 1$.

(3) If $r_n>0$ and $\{r_n\}$ converge increasingly to $r\in(0,1]$, then

$$\bigcap_{n=1}^{\infty}[\widetilde{A}]^{r_n}=[\widetilde{A}]^{r}. \tag{10}$$

Conversely, if for any $r\in[0,1]$, there exists $B_r\subset\mathbb{R}$ satisfying (1)–(3), then there exists a unique $\tilde{A}\in\tilde{E}$ such that $[\tilde{A}]^{r}=A^{r}$, $r\in(0,1]$ and $[\widetilde{A}]^{0}=\overline{\cup_{r\in(0,1]}[\widetilde{A}]^{r}}\subset B_0$.

Definition 3 (see [24]). A triangle fuzzy number $\tilde{A}$ is a fuzzy number with piecewise linear membership function $\tilde{A}$ defined by

$$u_{\widetilde{A}}(x)=\begin{cases}\dfrac{x-a_l}{a_m-a_l}, & a_l\leq x\leq a_m,\\[2mm] \dfrac{a_n-x}{a_n-a_m}, & a_m<x\leq a_n,\\[2mm] 0, & \text{otherwise},\end{cases} \tag{11}$$

which can be indicated as a triplet (a_l,a_m,a_n).

Given any two triangle fuzzy numbers $\widetilde{x}_i=(x_i-\delta_{i,1},x_i,x_i+\delta_{i,1})$ and $\widetilde{x}_j=(x_j-\delta_{j,1},x_j,x_j+\delta_{j,1}))$ and $k\geq 0$, the operational rules are as follows:

(1) $\widetilde{x}_i+\widetilde{x}_j=(x_i-\delta_{i,1}+x_j-\delta_{j,1},x_i+x_j,x_i+\delta_{i,1}+x_j+\delta_{j,1})$

(2) $k\cdot\widetilde{x}_i=(kx_i-k\delta_{i,1},kx_i,kx_i+k\delta_{i,2})$

Definition 4 (see [18]). Given a nonnegative matrix $P=[p_{ij}]$ and a fuzzy-number vector $\widetilde{X}$, if $P\in R_{+}^{m\times m}$ and $\widetilde{X}=[\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_m]^{T}\in\widetilde{E}^{m}$(the ^{T} denotes the conjugate transpose of a vector or a matrix.), then the product of P and X is defined as follows:

$$P\widetilde{X}_{n-1}=\begin{bmatrix}\sum_{j=1}^{m}p_{ij}\widetilde{x}_j\\ \vdots\\ \sum_{j=1}^{m}p_{mj}\widetilde{x}_j\end{bmatrix}. \tag{12}$$

3. Weighted Moving Averages for Fuzzy Numbers Based on a Nonadditive Measure with $\sigma-\lambda$ Rules and Choquet Integral of Fuzzy-Number-Valued Function

Definition 5 (see [18]). Let $(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_m)\in\widetilde{E}^{m}$, $(t_1,t_2,\ldots,t_m)\in R^{m}$, and g_λ be fuzzy measures satisfying $\delta-\lambda$ rules. Denote $A_i=\{t_i,t_{i+1},\ldots,t_m\}$, $i=1,2,\ldots,m$, and $A_{m+1}=\varnothing$. Then, the weighted moving averages for fuzzy numbers based on a nonadditive measure with $\sigma-\lambda$ rules is defined as follows:

$$\begin{aligned}\widetilde{x}_n&=(g_\lambda(A_1)-g_\lambda(A_2))\widetilde{x}_{n-m}+(g_\lambda(A_2)-g_\lambda(A_3))\widetilde{x}_{n-m+1}\\ &\quad+\cdots+(g_\lambda(A_m)-g_\lambda(A_{m+1}))\widetilde{x}_{n-1},\end{aligned} \tag{13}$$

where $n>m$.

Definition 6. Let $(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_m)\in\widetilde{E}^{m}$, $(t_1,t_2,\ldots,t_m)\in R^{m}$, and g_λ be fuzzy measures satisfying $\delta-\lambda$ rules. Let $A_i=\{t_i,t_{i+1},\ldots,t_m\}$, $i=1,2,\ldots,m$, and $A_{m+1}=\varnothing$. Then, for fuzzy number $\widetilde{x}_n(n>m)$, the Choquet integral of $\widetilde{x}_n(n>m)$ with respect to fuzzy measure g_λ on A is defined as follows:

$$(C)\int_A\widetilde{x}_n\mathrm{d}g_\lambda=\sum_{i=1}^{m}\widetilde{x}_{n-m+i-1}(g_\lambda(A_i)-g_\lambda(A_{i+1})). \tag{14}$$

Similarly, for vector $\widetilde{X}_n=[\widetilde{x}_n,\widetilde{x}_{n+1},\ldots,\widetilde{x}_{n+m-1}]^{T}$ $(n>m)$, the Choquet integral of $\widetilde{X}_n$ with respect to fuzzy measure g_λ on A is defined as follows:

$$\begin{aligned}(C)\int_A\widetilde{X}_n\mathrm{d}g_\lambda&=\Big[(C)\int_A\widetilde{x}_n\mathrm{d}g_\lambda,(C)\int_A\widetilde{x}_{n+1}\mathrm{d}g_\lambda,\ldots,(C)\\ &\quad\cdot\int_A\widetilde{x}_{n+m-1}\mathrm{d}g_\lambda\Big]^{T}.\end{aligned} \tag{15}$$

Remark 4. Accordingly, if $\widetilde{x}_n$ is a triangle fuzzy number, then the Choquet integral of fuzzy number $\widetilde{x}_n(n>m)$ with respect to fuzzy measure g_λ on A is defined as follows:

$$(C)\int_A \tilde{x}_n \mathrm{d}g_\lambda = \sum_{i=1}^{m}\left(\left(x_{n-m+i-1}-\delta_{n-m+i-1,1}\right)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right), x_{n-m+i-1}\left(\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right), \left(x_{n-m+i-1}+\delta_{n-m+i-1,2}\right)\right.\right.$$
$$\left.\cdot\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right)\right)\right)$$
$$=\left(\sum_{i=1}^{m}\left(x_{n-m+i-1}-\delta_{n-m+i-1,1}\right)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right),\right.$$
$$\sum_{i=1}^{m} x_{n-m+i-1}\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right),$$
$$\left.\sum_{i=1}^{m}\left(x_{n-m+i-1}+\delta_{n-m+i-1,2}\right)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right)\right), \tag{16}$$

where $\tilde{x}_n = (x_n - \delta_{n,1}, x_n, x_n + \delta_{n,2})$.

Theorem 1. *Let* $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \tilde{E}^m$, $(t_1, t_2, \ldots, t_m) \in R^m$, *and* g_λ *be fuzzy measures satisfying* $\delta-\lambda$ *rules. Denote* $A_i = \{t_i, t_{i+1}, \ldots, t_m\}$, $i = 1, 2, \ldots, m$, $A_{m+1} = \varnothing$, *and* t *be a positive real number. Then, for vector* $\tilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m-1}]^T$ $(n > m)$ *and*

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ g_\lambda(A_1)-g_\lambda(A_2) & g_\lambda(A_2)-g_\lambda(A_3) & g_\lambda(A_3)-g_\lambda(A_4) & \cdots & g_\lambda(A_m)-g_\lambda(A_{m+1}) \end{bmatrix}, \tag{17}$$

we have

(1)

$$(C)\int_A \tilde{X}_n \mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m} \tilde{x}_{n-m+i-1}\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^{m} \tilde{x}_{n-m+i}\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^{m} \tilde{x}_{n-m+i+1}\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \cdots \\ \sum_{i=1}^{m} \tilde{x}_{n+i-2}\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \end{bmatrix}. \tag{18}$$

(2)

$$(C)\int_A \tilde{X}_{n+1} \mathrm{d}g_\lambda = \mathbf{P}\cdot(C)\int_A \tilde{X}_n \mathrm{d}g_\lambda = \mathbf{P}^2\cdot$$
$$(C)\int_A \tilde{X}_{n-1} \mathrm{d}g_\lambda = \cdots = \mathbf{P}^{n-m-1}\cdot \tag{19}$$
$$(C)\int_A \tilde{X}_{m+2} \mathrm{d}g_\lambda = \mathbf{P}^{n-m}\cdot(C)\int_A \tilde{X}_{m+1} \mathrm{d}g_\lambda.$$

(3)

$$(C)\int_A \tilde{X}_{n+t} \mathrm{d}g_\lambda = \mathbf{P}^t\cdot(C)\int_A \tilde{X}_n \mathrm{d}g_\lambda, \tag{20}$$

especially, if $g_\lambda(A_1) - g_\lambda(A_2) > 0$ *and* $n - t > m$, *then*

$$(C)\int_A \tilde{X}_{n-t} \mathrm{d}g_\lambda = \mathbf{P}^{-t}\cdot(C)\int_A \tilde{X}_n \mathrm{d}g_\lambda. \tag{21}$$

(4) If gcd $\{i \in \{1, 2, \ldots, m\}: g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1$, *then* $\lim_{n\longrightarrow\infty}(C)\int_A \tilde{X}_n dg_\lambda$ *exists and*

$$\lim_{n\longrightarrow\infty}(C)\int_A \tilde{X}_n \mathrm{d}g_\lambda = \lim_{n\longrightarrow\infty}\mathbf{P}^{n-1}\cdot(C)\int_A \tilde{X}_{m+1} \mathrm{d}g_\lambda$$
$$= \frac{ea^T}{a^Te}\cdot(C)\int_A \tilde{X}_1 \mathrm{d}g_\lambda = er^{\mathrm{T}}\cdot(C)\int_A \tilde{X}_1 \mathrm{d}g_\lambda, \tag{22}$$

where $e = \sum_{i=1}^{m} e_k = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m\times 1}$ *and* e_k *is the ith standard unit column vector:*

$$\begin{aligned} a &= [a_1, a_2, \ldots, a_m]^{\mathrm{T}}, \\ b &= [b_1, b_2, \ldots, b_m]^{\mathrm{T}}, \\ a_k &= \sum_{i=1}^{k} (g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ b_k &= \frac{a^{\mathrm{T}} e_k}{a^{\mathrm{T}} e} = \frac{a_k}{\sum_{i=1}^{m} a_i} = \frac{g_\lambda(A_1) - g_\lambda(A_{k+1})}{m g_\lambda(A_1) - \sum_{i=2}^{m} g_\lambda(A_i)}, \quad k = 1, 2, 3, \ldots, m. \end{aligned} \tag{23}$$

Proof

(1) According to Definition 6, we know that

$$\begin{aligned} (C)\int_A \widetilde{x}_n \mathrm{d}g_\lambda &= \sum_{i=1}^{m} \widetilde{x}_{n-m+i-1} (g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ (C)\int_A \widetilde{x}_{n+1} \mathrm{d}g_\lambda &= \sum_{i=1}^{m} \widetilde{x}_{n-m+i} (g_\lambda(A_i) - g_\lambda(A_{i+1})), \ldots, \\ (C)\int_A \widetilde{x}_{n+m-1} \mathrm{d}g_\lambda &= \sum_{i=1}^{m} \widetilde{x}_{n+i-2} (g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda &= \left[(C)\int_A \widetilde{x}_n \mathrm{d}g_\lambda, (C)\int_A \widetilde{x}_{n+1} \mathrm{d}g_\lambda, \ldots, (C)\int_A \widetilde{x}_{n+m-1} \mathrm{d}g_\lambda\right]^{\mathrm{T}}. \end{aligned} \tag{24}$$

Thus, we have

$$(C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m} \widetilde{x}_{n-m+i-1} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \sum_{i=1}^{m} \widetilde{x}_{n-m+i} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \sum_{i=1}^{m} \widetilde{x}_{n-m+i+1} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \cdots \\ \sum_{i=1}^{m} \widetilde{x}_{n+i-2} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \end{bmatrix}. \tag{25}$$

(2) According to Definition 6, we can obtain

$$\mathbf{P} \cdot (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m} \widetilde{x}_{n-m+i} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \sum_{i=1}^{m} \widetilde{x}_{n-m+i+1} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \sum_{i=1}^{m} \widetilde{x}_{n-m+i+2} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \\ \cdots \\ \sum_{i=1}^{m} \widetilde{x}_{n+i-1} (g_\lambda(A_i) - g_\lambda(A_{i+1})) \end{bmatrix}. \tag{26}$$

Then, by the expression of $(C)\int_A \widetilde{X}_{n+1} \mathrm{d}g_\lambda$ in (1), we have

$$(C)\int_A \widetilde{x}_{n+1} \mathrm{d}g_\lambda = \mathbf{P} \cdot (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda. \tag{27}$$

(3) By (2), we know that

$$(C)\int_A \widetilde{X}_{n+t} \mathrm{d}g_\lambda = \mathbf{P}^t \cdot (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda. \tag{28}$$

Since $\mathbf{P}$ is an invertible matrix, we have

$$(C)\int_A \widetilde{X}_{n-t} \mathrm{d}g_\lambda = \mathbf{P}^{-t} \cdot (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda. \tag{29}$$

(4) By using Theorem 2 in Reference [18], we note that $\lim_{n \longrightarrow \infty} \mathbf{P}^n$ exists and

$$\lim_{n \longrightarrow \infty} \mathbf{P}^n = \frac{e a^{\mathrm{T}}}{a^{\mathrm{T}} e} = e r^{\mathrm{T}}. \tag{30}$$

Combining (3), it follows that

$$(C)\int_A \widetilde{X}_{n+t} \mathrm{d}g_\lambda = \mathbf{P}^t \cdot (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda. \tag{31}$$

Take limit of the above equation, we obtain

$$\begin{aligned} \lim_{n \longrightarrow \infty} (C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda &= \lim_{n \longrightarrow \infty} \mathbf{P}^{n-1} \cdot (C)\int_A \widetilde{X}_1 \mathrm{d}g_\lambda = \frac{e a^{\mathrm{T}}}{a^{\mathrm{T}} e} \cdot \\ (C)\int_A \widetilde{X}_1 \mathrm{d}g_\lambda &= e r^{\mathrm{T}} \cdot (C)\int_A \widetilde{X}_1 \mathrm{d}g_\lambda. \end{aligned} \tag{32}$$

The proof is complete. □

Definition 7. For vector $\widetilde{X}_n = [\widetilde{x}_n, \widetilde{x}_{n+1}, \ldots, \widetilde{x}_{n+m-1}]^{\mathrm{T}}$ $(n > m)$ and

$$\begin{aligned} \widetilde{X}_n^-(r) &= [\widetilde{x}_n^-(r), \widetilde{x}_{n+1}^-(r), \ldots, \widetilde{x}_{n+m-1}^-(r)]^{\mathrm{T}}, \\ \widetilde{X}_n^+(r) &= [\widetilde{x}_n^+(r), \widetilde{x}_{n+1}^+(r), \ldots, \widetilde{x}_{n+m-1}^+(r)]^{\mathrm{T}}, \end{aligned} \tag{33}$$

the Choquet integral of $\widetilde{X}_n^-(r)$ with respect to fuzzy measure g_λ on A is defined as follows:

$$\begin{aligned} (C)\int_A \widetilde{X}_n^-(r) \mathrm{d}g_\lambda = \Big[(C)\int_A \widetilde{x}_n^-(r) \mathrm{d}g_\lambda, (C)\int_A \widetilde{x}_{n+1}^-(r) \mathrm{d}g_\lambda, \ldots, \\ \cdot (C)\int_A \widetilde{x}_{n+m-1}^-(r) \mathrm{d}g_\lambda\Big]^{\mathrm{T}}. \end{aligned} \tag{34}$$

Also, the Choquet integral of $\widetilde{X}_n^-(r)$ with respect to fuzzy measure g_λ on A is defined by

$$(C)\int_A \widetilde{X}_n^+(r)\mathrm{d}g_\lambda = \left[(C)\int_A \widetilde{x}_n^+(r)\mathrm{d}g_\lambda, (C)\int_A \widetilde{x}_{n+1}^+(r)\mathrm{d}g_\lambda, \ldots, \cdot (C)\int_A \widetilde{x}_{n+m-1}^+(r)\mathrm{d}g_\lambda\right]^T. \quad (35)$$

Theorem 2. *Let* $(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_m) \in \widetilde{E}^m$, $(t_1, t_2, \ldots, t_m) \in R^m$, *and* g_λ *be the fuzzy measure satisfying* $\delta-\lambda$ *rules. Denote* $A_i = \{t_i, t_{i+1}, \ldots, t_m\}$, $i = 1, 2, \ldots, m$, *and* $A_{m+1} = \varnothing$, *and t be the positive real number. Then, for vector*

$$\widetilde{X}_n^-(r) = \left[\widetilde{x}_n^-(r), \widetilde{x}_{n+1}^-(r), \ldots, \widetilde{x}_{n+m-1}^-(r)\right]^{\mathrm{T}}, \quad (36)$$

where $\mathbf{P}$ *is the same matrix in Theorem 1, we have*

(1)

$$(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^m \widetilde{x}_{n-m+i-1}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i+1}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right) \\ \cdots \\ \sum_{i=1}^m \widetilde{x}_{n+i-2}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right) \end{bmatrix}. \quad (37)$$

(2)

$$\begin{aligned} rpt\,(C)\int_A \widetilde{X}_{n+1}^-(r)\mathrm{d}g_\lambda &= \mathbf{P}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \mathbf{P}^2\cdot (C)\int_A \widetilde{X}_{n-1}^-(r)\mathrm{d}g_\lambda = \cdots \\ &= \mathbf{P}^{n-m-1}\cdot(C)\int_A \widetilde{X}_{m+2}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^{n-m}\cdot(C)\int_A \widetilde{X}_{m+1}^-(r)\mathrm{d}g_\lambda. \end{aligned} \quad (38)$$

(3)

$$(C)\int_A \widetilde{X}_{n+t}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^t\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda, \quad (39)$$

especially, if $g_\lambda(A_1) - g_\lambda(A_2) > 0$ *and* $n - t > m$, *then*

$$(C)\int_A \widetilde{X}_{n-t}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^{-t}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda. \quad (40)$$

(4) If $\gcd\{i \in \{1, 2, \ldots, m\}: g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1$, *then* $\lim_{n\longrightarrow\infty}(C)\int_A \widetilde{X}_n^-(r)dg_\lambda$ *exists and*

$$\begin{aligned} \lim_{n\longrightarrow\infty}(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda &= \lim_{n\longrightarrow\infty}\mathbf{P}^{n-1}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda \\ &= \frac{ea^{\mathrm{T}}}{a^{\mathrm{T}}e}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda = eb^{\mathrm{T}}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda, \end{aligned} \quad (41)$$

where $e = \sum_{i=1}^m e_k = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m\times 1}$ *and* e_k *is the ith standard unit column vector:*

$$\begin{aligned} a &= [a_1, a_2, \ldots, a_m]^{\mathrm{T}}, \\ b &= [b_1, b_2, \ldots, b_m]^{\mathrm{T}}, \\ a_k &= \sum_{i=1}^k \left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \\ b_k &= \frac{a^{\mathrm{T}}e_k}{a^{\mathrm{T}}e} = \frac{a_k}{\sum_{i=1}^m a_i} = \frac{g_\lambda(A_1) - g_\lambda(A_{k+1})}{mg_\lambda(A_1) - \sum_{i=2}^m g_\lambda(A_i)}, \quad k = 1, 2, 3, \ldots, m. \end{aligned} \quad (42)$$

Proof

(1) According to Definition 6, we know that

$$\begin{aligned} (C)\int_A \widetilde{x}_n^-(r)\mathrm{d}g_\lambda &= \sum_{i=1}^m \widetilde{x}_{n-m+i-1}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \\ (C)\int_A \widetilde{x}_{n+1}^-(r)\mathrm{d}g_\lambda &= \sum_{i=1}^m \widetilde{x}_{n-m+i}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \ldots, \\ (C)\int_A \widetilde{x}_{n+m-1}^-(r)\mathrm{d}g_\lambda &= \sum_{i=1}^m \widetilde{x}_{n+i-2}^-(r)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right). \end{aligned} \quad (43)$$

Furthermore,

$$(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \left[(C)\int_A \widetilde{x}_n^-(r)\mathrm{d}g_\lambda, (C)\int_A \widetilde{x}_{n+1}^-(r)\mathrm{d}g_\lambda, \ldots, \cdot (C)\int_A \widetilde{x}_{n+m-1}^-(r)\mathrm{d}g_\lambda\right]^{\mathrm{T}}. \quad (44)$$

Thus, we have

$$(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^m \widetilde{x}_{n-m+i-1}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i+1}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \cdots \\ \sum_{i=1}^m \widetilde{x}_{n+i-2}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \end{bmatrix}. \tag{45}$$

(2) According to Definition 7, we can obtain

$$\mathbf{P}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^m \widetilde{x}_{n-m+i}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i+1}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i+1+2}^-(r)\left(g_\lambda(A_i))-g_\lambda(A_{i+1})\right) \\ \cdots \\ \sum_{i=1}^m \widetilde{x}_{n+i-1}^-(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \end{bmatrix}. \tag{46}$$

Then, by the expression of $(C)\int_A \widetilde{X}_{n+1}^-(r)\mathrm{d}g_\lambda$ in (1), we have

$$(C)\int_A \widetilde{X}_{n+1}^-(r)\mathrm{d}g_\lambda = \mathbf{P}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda. \tag{47}$$

(3) By (2), we know that

$$(C)\int_A \widetilde{X}_{n+t}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^t\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda. \tag{48}$$

Since $\mathbf{P}$ is an invertible matrix, we have

$$(C)\int_A \widetilde{X}_{n-t}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^{-t}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda. \tag{49}$$

(4) By using Theorem 2 in Reference [18], we note that $\lim_{n\longrightarrow\infty}\mathbf{P}^n$ exists and

$$\lim_{n\longrightarrow\infty}\mathbf{P}^n = \frac{ea^{\mathrm{T}}}{a^{\mathrm{T}}e} = er^{\mathrm{T}}. \tag{50}$$

Combining (3), it follows that

$$(C)\int_A \widetilde{X}_{n+t}^-(r)\mathrm{d}g_\lambda = \mathbf{P}^t\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda. \tag{51}$$

Taking limit of the above equation, we obtain

$$\lim_{n\longrightarrow\infty}(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \lim_{n\longrightarrow\infty}\mathbf{P}^{n-1}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda$$
$$= \frac{ea^{\mathrm{T}}}{a^{\mathrm{T}}e}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda = eb^{\mathrm{T}}\cdot(C)\int_A \widetilde{X}_1^-(r)\mathrm{d}g_\lambda. \tag{52}$$

The proof is complete. □

Theorem 3. *Let $(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_m)\in\widetilde{E}^m$, $(t_1, t_2, \ldots, t_m)\in R^m$, and g_λ be a fuzzy measure satisfying $\delta-\lambda$ rules. Denote $A_i=\{t_i, t_{i+1}, \ldots, t_m\}$, $i=1, 2, \ldots, m$, and $A_{m+1}=\varnothing$, and t be a positive real number. For vector*

$$\widetilde{X}_n^+(r) = \left[\widetilde{x}_n^+(r), \widetilde{x}_{n+1}^+(r), \ldots, \widetilde{x}_{n+m-1}^+(r)\right]^{\mathrm{T}}, \tag{53}$$

we have

(1)

$$(C)\int_A \widetilde{X}_n^+(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^m \widetilde{x}_{n-m+i-1}^+(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i}^+(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \sum_{i=1}^m \widetilde{x}_{n-m+i+1}^+(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \\ \cdots \\ \sum_{i=1}^m \widetilde{x}_{n+i-2}^+(r)\left(g_\lambda(A_i)-g_\lambda(A_{i+1})\right) \end{bmatrix}. \tag{54}$$

(2)

$$(C)\int_A \widetilde{X}_{n+1}^-(r)\mathrm{d}g_\lambda = \mathbf{P}\cdot(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \mathbf{P}^2\cdot$$
$$(C)\int_A \widetilde{X}_{n-1}^+(r)\mathrm{d}g_\lambda = \cdots$$
$$= \mathbf{P}^{n-m-1}\cdot(C)\int_A \widetilde{X}_{m+2}^+(r)\mathrm{d}g_\lambda = \mathbf{P}^{n-m}\cdot(C)\int_A \widetilde{X}_{m+1}^+(r)\mathrm{d}g_\lambda. \tag{55}$$

(3)

$$(C)\int_A \widetilde{X}_{n+t}^+(r)\mathrm{d}g_\lambda = \mathbf{P}^t\cdot(C)\int_A \widetilde{X}_n^+(r)\mathrm{d}g_\lambda, \tag{56}$$

especially, if $g_\lambda(A_1)-g_\lambda(A_2)>0$ and $n-t>m$, then

$$(C)\int_A \widetilde{X}_{n-t}^{+}(r)\mathrm{d}g_\lambda = \mathbf{P}^{-t}\cdot (C)\int_A \widetilde{X}_{n}^{+}(r)\mathrm{d}g_\lambda. \quad (57)$$

(4) If gcd $\{i \in \{1, 2, \ldots, m\}: g_\lambda(A_i) - g_\lambda(A_{i+1}) > 0\} = 1$, then $\lim_{n\longrightarrow\infty}(C)\int_A \widetilde{X}_n^{+}(r)dg_\lambda$ exists and

$$\begin{aligned}\lim_{n\longrightarrow\infty}(C)\int_A \widetilde{X}_{n}^{+}(r)\mathrm{d}g_\lambda &= \lim_{n\longrightarrow\infty}\mathbf{P}^{n}\cdot (C)\int_A \widetilde{X}_{m+1}^{+}(r)\mathrm{d}g_\lambda \\ &= \frac{ea^{\mathrm{T}}}{a^{\mathrm{T}}e}\cdot (C)\int_A \widetilde{X}_{m+1}^{+}(r)\mathrm{d}g_\lambda = eb^{\mathrm{T}}\cdot (C)\int_A \widetilde{X}_{m+1}^{+}(r)\mathrm{d}g_\lambda.\end{aligned} \quad (58)$$

where $e = \sum_{i=1}^{m} e_k = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m\times 1}$ and e_k is the ith standard unit column vector:

$$\begin{aligned} a &= [a_1, a_2, \ldots, a_m]^{\mathrm{T}}, \\ b &= [b_1, b_2, \ldots, b_m]^{\mathrm{T}}, \\ a_k &= \sum_{i=1}^{k}(g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ b_k &= \frac{a^{\mathrm{T}}e_k}{a^{\mathrm{T}}e} = \frac{a_k}{\sum_{i=1}^{m} a_i} = \frac{g_\lambda(A_1) - g_\lambda(A_{k+1})}{mg_\lambda(A_1) - \sum_{i=2}^{m} g_\lambda(A_i)}, \quad k = 1, 2, 3, \ldots, m. \end{aligned} \quad (59)$$

Proof. Theorem 1 implies. □

Theorem 4. *Let $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \in \widetilde{E}^m$, $(t_1, t_2, \ldots, t_m) \in R^m$, and g_λ be a fuzzy measure satisfying $\delta-\lambda$ rules. Denote $A_i = \{t_i, t_{i+1}, \ldots, t_m\}$, $i = 1, 2, \ldots, m$, and $A_{m+1} = \varnothing$, and t be a positive real number. If $\tilde{x}_i$ is a triangle fuzzy number, and $\tilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \ldots,$ then for*

$$\widetilde{X}_n = [\tilde{x}_n, \tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m-1}]^{\mathrm{T}}, \quad (60)$$

we have

(1)

$$\begin{aligned}(C)\int_A \widetilde{X}_n \mathrm{d}g_\lambda = \Bigg[&\Bigg(\sum_{i=1}^{m}\left(x_{n-m+i-1} - \delta_{n-m+i-1,1}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ &\sum_{i=1}^{m} x_{n-m+i-1}(g_\lambda(A_i) - g_\lambda(A_{i+1})), \sum_{i=1}^{m}\left(x_{n-m+i-1} + \delta_{n-m+i-1,2}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1}))\Bigg), \\ &\cdot\Bigg(\sum_{i=1}^{m}\left(x_{n-m+i} - \delta_{n-m+i,1}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ &\sum_{i=1}^{m} x_{n-m+i}(g_\lambda(A_i) - g_\lambda(A_{i+1})), \sum_{i=1}^{m}\left(x_{n-m+i} + \delta_{n-m+i,2}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1}))\Bigg), \\ &\ldots, \\ &\cdot\Bigg(\sum_{i=1}^{m}\left(x_{n+i-2} - \delta_{n+i-2,1}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1})), \\ &\sum_{i=1}^{m} x_{n+i-2}(g_\lambda(A_i) - g_\lambda(A_{i+1})), \sum_{i=1}^{m}\left(x_{n+i-2} + \delta_{n+i-2,2}\right)(g_\lambda(A_i) - g_\lambda(A_{i+1}))\Bigg)\Bigg]^{\mathrm{T}}.\end{aligned} \quad (61)$$

(2)

$$(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m}\left(\delta_{n-m+i-1,1}r + x_{n-m+i-1} - \delta_{n-m+i-1,1}\right) \\ \sum_{i=1}^{m}\left(\delta_{n-m+i,1}r + x_{n-m+i} - \delta_{n-m+i,1}\right) \\ \sum_{i=1}^{m}\left(\delta_{n-m+i+1,1}r + x_{n-m+i+1} - \delta_{n-m+i+1,1}\right) \\ \cdots \\ \sum_{i=1}^{m}\left(\delta_{n+i-2,1}r + x_{n+i-2} - \delta_{n+i-2,1}\right) \end{bmatrix}. \quad (62)$$

(3)

$$(C)\int_A \widetilde{X}_n^+(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m}\left(-\delta_{n-m+i-1,2}r + x_{n-m+i-1} + \delta_{n-m+i-1,2}\right) \\ \sum_{i=1}^{m}\left(-\delta_{n-m+i,2}r + x_{n-m+i} + \delta_{n-m+i,2}\right) \\ \sum_{i=1}^{m}\left(-\delta_{n-m+i+1,2}r + x_{n-m+i+1} + \delta_{n-m+i+1,2}\right) \\ \cdots \\ \sum_{i=1}^{m}\left(-\delta_{n+i-2,2}r + x_{n+i-2} + \delta_{n+i-2,2}\right) \end{bmatrix}. \quad (63)$$

Proof

(1) By Remark 4, we know that the Choquet integral of fuzzy number $\widetilde{x}_n (n > m)$ with respect to fuzzy measure g_λ on A of is

$$(C)\int_A \widetilde{x}_n \mathrm{d}g_\lambda = \left(\sum_{i=1}^{m}\left(x_{n-m+i-1} - \delta_{n-m+i-1,1}\right)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \sum_{i=1}^{m} x_{n-m+i-1}\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right), \sum_{i=1}^{m}\left(x_{n-m+i-1} + \delta_{n-m+i-1,2}\right)\left(g_\lambda(A_i) - g_\lambda(A_{i+1})\right)\right), \quad (64)$$

where $\widetilde{x}_n = (\widetilde{x}_n - \delta_{n,1}, \widetilde{x}_n, \widetilde{x}_n + \delta_{n,2})$. Combining Theorem 1 (1), we can obtain the above equation.

(2) According to Theorem 3 in Reference [18], we know that if $\widetilde{x}_i$ is a triangle fuzzy number and $\widetilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \ldots,$ then

$$\begin{aligned}\widetilde{X}_n^-(r) &= \left[\widetilde{x}_n^-(r), \widetilde{x}_{n+1}^-(r), \ldots, \widetilde{x}_{n+m-1}^-(r)\right]^{\mathrm{T}} \\ &= \left[\delta_{n,1}r + x_n - \delta_{n,1}, \delta_{n+1,1}r + x_{n+1} - \delta_{n+1,1}, \ldots, \delta_{n+m-1,1}r + x_{n+m-1} - \delta_{n+m-1,1}\right]^{\mathrm{T}},\end{aligned} \quad (65)$$

and by Remark 4 and Theorem 3, we have

$$(C)\int_A \widetilde{X}_n^-(r)\mathrm{d}g_\lambda = \begin{bmatrix} \sum_{i=1}^{m}\left(\delta_{n-m+i-1,1}r + x_{n-m+i-1} - \delta_{n-m+i-1,1}\right) \\ \sum_{i=1}^{m}\left(\delta_{n-m+i,1}r + x_{n-m+i} - \delta_{n-m+i,1}\right) \\ \sum_{i=1}^{m}\left(\delta_{n-m+i+1,1}r + x_{n-m+i+1} - \delta_{n-m+i+1,1}\right) \\ \cdots \\ \sum_{i=1}^{m}\left(\delta_{n+i-2,1}r + x_{n+i-2} - \delta_{n+i-2,1}\right) \end{bmatrix}. \quad (66)$$

(3) (2) implies.

The proof is complete. □

Example 1. We choose the same example in Reference [18] to illustrate our study and make comparison. Given a closing stock price system over 5 days, the closing prices of each day are denoted as $\widetilde{x}_i$, $(\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_5) \in \widetilde{E}^5$, and every $\widetilde{x}_i$ is a triangle fuzzy number, $\widetilde{x}_i = (x_i - \delta_{i,1}, x_i, x_i + \delta_{i,2})$, $i = 1, 2, \ldots, 5$. Suppose $(t_1, t_2, \ldots, t_5) \in R^5$, $A_i = \{t_i, t_{i+1}, \ldots, t_5\}$, $i = 1, 2, \ldots, 5$, and $A_6 = \varnothing$. The value and weight of each $\widetilde{x}_i$, $i = 1, 2, \ldots, 5$, are shown in Table 1. Then, we can obtain the closing stock price over 10 days and some relevant results.

According to Remark 3 in Reference [18], we can obtain

$$\begin{aligned} g_\lambda(A_1) &= 1, \\ g_\lambda(A_2) &= 0.88, \\ g_\lambda(A_3) &= 0.65, \\ g_\lambda(A_4) &= 0.33, \\ g_\lambda(A_5) &= 0.175, \\ g_\lambda(A_6) &= 0. \end{aligned} \quad (67)$$

By Definition 6 and Remark 4, the Choquet integral of $\widetilde{x}_6$ with respect to fuzzy measure g_λ on A is determined as follows:

$$\widetilde{x}_6 = (C)\int_A \widetilde{x}_6 \mathrm{d}g_\lambda = (22.04, 23.04, 24.04). \quad (68)$$

Similarly, we can also calculate the Choquet integral of $\widetilde{x}_n$, $n = 7, 8, 9, 10$, with respect to fuzzy measure g_λ on A, as shown in Table 2.

And according to Definition 6 and Theorem 4, the Choquet integral of fuzzy number vector $\widetilde{X}_6 = [\widetilde{x}_6, \widetilde{x}_7, \ldots, \widetilde{x}_{10}]^{\mathrm{T}}$ with respect to fuzzy measure g_λ on A is determined as follows:

$$\begin{aligned}(C)\int_A \widetilde{X}_6 \mathrm{d}g_\lambda &= \left[(C)\int_A \widetilde{x}_6 \mathrm{d}g_\lambda, (C)\int_A \widetilde{x}_7 \mathrm{d}g_\lambda, \ldots, (C)\int_A \widetilde{x}_{10} \mathrm{d}g_\lambda\right]^{\mathrm{T}} \\ &= [(22.04, 23.04, 24.04), (22.76, 23.76, 24.76), \\ &\cdot (22.72, 23.72, 24.72), \\ &(22.5, 23.5, 24.5), (22.45, 23.45, 24.45)]^{\mathrm{T}}.\end{aligned} \quad (69)$$

Table 1: Closing stock prices over 5 days.

Day	Closing stock price	g_λ
1	(19, 20, 21)	0.1
2	(21, 22, 23)	0.2
3	(23, 24, 25)	0.3
4	(24, 25, 26)	0.15
5	(22, 23, 24)	0.175

Table 2: Closing stock prices over 10 days.

Day	Closing stock price	g_λ
1	(19, 20, 21)	0.1
2	(21, 22, 23)	0.2
3	(23, 24, 25)	0.3
4	(24, 25, 26)	0.15
5	(22, 23, 24)	0.175
6	(22.04, 23.04, 24.04)	
7	(22.76, 23.76, 24.76)	
8	(22.72, 23, 72, 24.72)	
3 9	(22.5, 23.5, 24.5)	
10	(22.45, 23.45, 24.45)	

This article is a complement of our previous work [18]; namely, the method presented in this article can be regarded as a generalization of the previous method [18]. That is, the calculation of the moving average for a series of fuzzy numbers in [18] is transformed into Choquet integration of fuzzy-number-valued function under discrete case in this work. More specifically, compared with our previous work in Reference [18], we introduce the new concepts: the Choquet integral of fuzzy number and the Choquet integral of fuzzy number vector, containing m elements needed to make forecasting of the $m + 1_{\text{th}}$ element. These new concepts provide a possibility to dealing with the moving average from vector integral, which could describe the moving average of time series in a more intuitive perspective using an important mathematical tool.

Meanwhile, when the data degenerate into distinct data and the nonadditive measure degenerates into probability measure, our method will degenerate into the classical moving weighted average method. Therefore, this method is the extension of the classical method. In this paper, we consider the mutual influence and connection of time nodes, while in the classical method, time nodes are independent of each other. Moreover, the classical time series cannot deal with problems of natural language assignment, Internet language assignment, qualitative description, etc. So, the advantage of this method is obvious.

4. Conclusion

In this paper, on the combination of Choquet integral and fuzzy number, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined. And it shows that the calculation of the moving average for a series of fuzzy numbers can be transformed into Choquet integration of fuzzy-number-valued function under discrete case. Subsequently, the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are defined, respectively. Finally, by means of the convolution formula of Choquet integral, some properties of the Choquet integral of fuzzy number and Choquet integral of fuzzy number vector are also investigated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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