Some Results for Pythagorean Fuzzy Sets

Xindong Peng, Yong Yang*
College of Computer Science and Engineering, Northwest Normal University, Lanzhou, People's Republic of China

Pythagorean fuzzy sets (PFSs), originally proposed by Yager (Yager, Abbasov. *Int J Intell Syst* 2013;28:436–452), are a new tool to deal with vagueness considering the membership grades are pairs (μ, ν) satisfying the condition $\mu^2 + \nu^2 \leq 1$. As a generalized set, PFSs have close relationship with intuitionistic fuzzy sets (IFSs). PFSs can be reduced to IFSs satisfying the condition $\mu + \nu \leq 1$. However, the related operations of PFSs do not take different conditions into consideration. To better understand PFSs, we propose two operations: division and subtraction, and discuss their properties in detail. Then, based on Pythagorean fuzzy aggregation operators, their properties such as boundedness, idempotency, and monotonicity are investigated. Later, we develop a Pythagorean fuzzy superiority and inferiority ranking method to solve uncertainty multiple attribute group decision making problem. Finally, an illustrative example for evaluating the Internet stocks performance is given to verify the developed approach and to demonstrate its practicality and effectiveness. © 2015 Wiley Periodicals, Inc.

1. INTRODUCTION

Atanassov[1] initiated the concept of intuitionistic fuzzy set (IFS), which is a generalization of Zadeh's fuzzy sets.[2] Each element in the IFS is expressed by an ordered pair (μ, ν) satisfying the condition $\mu + \nu \leq 1$. IFS has its greatest use in practical multiple attribute decision making (MADM) problems, and the academic research have achieved great development.[3–8]

However, in the some practical problems, the sum of membership degree and nonmembership degree to which an alternative satisfying an attribute provided by decision maker (DM) may be bigger than 1, but their square sum is less than or equal 1. Therefore, Yager[9–11] developed Pythagorean fuzzy set (PFS) characterized by a membership degree and nonmembership degree, which satisfies the condition that the square sum of its membership degree and nonmembership degree is less than or equal to 1. Yager[9] gave an example to state this situation: a DM gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his support against membership is $\frac{1}{2}$. Owing to the sum of two values is bigger than 1, they are not available for IFS, but they are available for PFS since $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \leq 1$. Obviously, PFS is more capable than IFS to model the vagueness in the practical problems.

*Author to whom all correspondence should be addressed; e-mail: yangzt@nwnu.edu.cn.

INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 30, 1133–1160 (2015)

View this article online at wileyonlinelibrary.com. • DOI 10.1002/int.21738

For further research of PFS, based on modificatory TOPSIS method,[12] Zhang and Xu[13] proposed an extension of Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) to solve MADM problem with Pythagorean fuzzy information. Yager[9,10] proposed four kinds of aggregation operators: Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric average (PFWG) operator, Pythagorean fuzzy weighted power average (PFWPA) operator, Pythagorean fuzzy weighted power geometric (PFWPG) operator, and applied them to MADM problems. However, few studies[13] focus on operations over PFS. Therefore, there is a need to define some new operations such as division, subtraction, and discuss their properties. Meanwhile, we explore some properties such as boundedness, idempotency, and monotonicity over Pythagorean fuzzy aggregation operators. In this paper, we will extend the superiority and inferiority ranking (SIR) method to solve the multiple attribute group decision making (MAGDM) problems with Pythagorean fuzzy information. The SIR method was first introduced by Xu[14] simultaneously employs the superiority and inferiority information to reflect the experts' attitude toward each attribute and describe the priority alternatives more comprehensively and accurately. It is a significant generalization of the well-known Preference Ranking Organisation MeTHod for Enrichment Evaluations (PROMETHEE) method.[15] Some scholars have extended the SIR method to solve the MAGDM problems with different fuzzy environments, such as in intuitionistic fuzzy environment[16,17] and hesitant fuzzy environment.[18] However, both SIR method and its extension fail to solve the MAGDM problems with Pythagorean fuzzy information. Therefore, we develop a novel SIR method named Pythagorean fuzzy SIR (PF-SIR).

The remainder of this paper is organized as follows. In Section 2, some basic definitions of IFS and PFS are briefly reviewed. In Section 3, two new operations division and subtraction are developed and their properties with others operations are discussed in detail. In Section 4, the relationship of Pythagorean fuzzy aggregation operators are investigated. In Section 5, we develop a PF-SIR method to solve MAGDM problem. In Section 6, a numerical example for evaluating Internet stocks is given to illustrate the proposed method. The paper is concluded in Section 7.

2. SOME BASIC CONCEPTS OF IFS AND PFS

In the following, some basic concepts related to IFS and PFS are introduced.

DEFINITION 1 ([1]). *Let X be a universe of discourse. An IFS I in X is given by*

$$I = \{< x, \mu_I(x), \nu_I(x) >| \, x \in X\}, \tag{1}$$

where $\mu_I : X \to [0,1]$ denotes the degree of membership and $\nu_I : X \to [0,1]$ denotes the degree of nonmembership of the element $x \in X$ to the set I, respectively, with the condition that $0 \le \mu_I(x) + \nu_I(x) \le 1$. The degree of indeterminacy $\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)$.

Yager[9–11] proposed a novel concept of PFS to model the condition that the sum of the degree to which an alternative x_i satisfies and dissatisfies with respect to the attribute C_j is bigger than 1, while the IFS cannot deal with it.

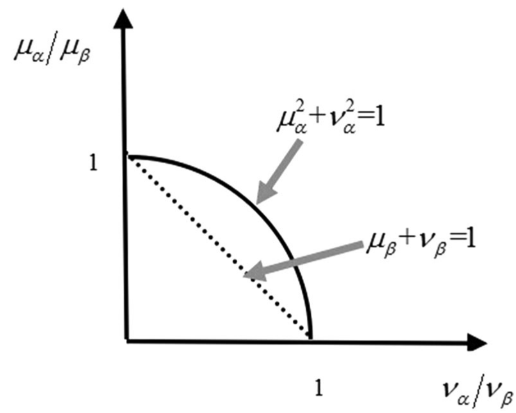

Figure 1. Comparison of spaces of the PFNs and IFNs.

DEFINITION 2 ([9–11]). *Let X be a universe of discourse. An PFS P in X is given by*

$$P = \{< x, \mu_P(x), \nu_P(x) >| \; x \in X\}, \tag{2}$$

where $\mu_P : X \to [0,1]$ *denotes the degree of membership and* $\nu_P : X \to [0,1]$ *denotes the degree of nonmembership of the element* $x \in X$ *to the set P, respectively, with the condition that* $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. *The degree of indeterminacy* $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$. *For convenience, Zhang and Xu*[13] *called* $(\mu_P(x), \nu_P(x))$ *a Pythagorean fuzzy number (PFN) denoted by* $p = (\mu_p, \nu_p)$.

Based on above definition, Zhang and Xu[13] *defined the distance between* p_1, p_2 *as follows:*

$$d(p_1, p_2) = \frac{1}{2}(|(\mu_{p_1})^2 - (\mu_{p_2})^2| + |(\nu_{p_1})^2 - (\nu_{p_2})^2| + |(\pi_{p_1})^2 - (\pi_{p_2})^2|) \tag{3}$$

The main difference between PFN and IFN is their corresponding constraint conditions, which is shown in Figure 1.[9]

DEFINITION 3 ([13]). *For any PFN* $p = (\mu_p, \nu_p)$, *the score function of p be defined as follows:*

$$s(p) = (\mu_p)^2 - (\nu_p)^2, \tag{4}$$

where $s(p) \in [-1, 1]$. *For any two PFNs* p_1, p_2, *if* $s(p_1) < s(p_2)$, *then* $p_1 \prec p_2$. *If* $s(p_1) > s(p_2)$, *then* $p_1 \succ p_2$. *If* $s(p_1) = s(p_2)$, *then* $p_1 \sim p_2$.

It is easily known that the score function defined in above is not unreasonable. For example, when two PFNs $p_1 = (0.5, 0.5)$ and $p_2 = (0.6, 0.6)$, based on Definition 3, $p_1 \sim p_2$. But in fact, it is not reasonable, so we propose the accuracy function and modify the comparison rules.

DEFINITION 4. *For any PFNs $p = (\mu_p, \nu_p)$, the accuracy function of p be defined as follows:*

$$a(p) = (\mu_p)^2 + (\nu_p)^2, \tag{5}$$

where $a(p) \in [0, 1]$.

For any two PFNs p_1, p_2,

(1) if $s(p_1) > s(p_2)$, then $p_1 \succ p_2$;
(2) if $s(p_1) = s(p_2)$, then
  (a) if $a(p_1) > a(p_2)$, then $p_1 \succ p_2$;
  (b) if $a(p_1) = a(p_2)$, then $p_1 \sim p_2$.

DEFINITION 5 ([9–11]). *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0$, then their operations are defined as follows:*

(1) $p_1 \cup p_2 = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\})$;
(2) $p_1 \cap p_2 = (\min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\})$;
(3) $p^c = (\nu, \mu)$.

DEFINITION 6 ([13]). *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0$, then their operations are defined as follows:*

(1) $p_1 \oplus p_2 = (\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2)$;
(2) $p_1 \otimes p_2 = (\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2})$;
(3) $\lambda p = (\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda)$;
(4) $p^\lambda = (\mu^\lambda, \sqrt{1 - (1 - \nu^2)^\lambda})$.

THEOREM 1 ([13]). *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then,*

(1) $p_1 \oplus p_2 = p_2 \oplus p_1$;
(2) $p_1 \otimes p_2 = p_2 \otimes p_1$;
(3) $\lambda(p_1 \oplus p_2) = \lambda p_1 \oplus \lambda p_2$;
(4) $\lambda_1 p \oplus \lambda_2 p = (\lambda_1 + \lambda_2)p$;
(5) $(p_1 \otimes p_2)^\lambda = p_1^\lambda \otimes p_2^\lambda$;
(6) $p^{\lambda_1} \otimes p^{\lambda_2} = p^{(\lambda_1+\lambda_2)}$.

3. SOME OPERATIONS AND PROPERTIES FOR PFNS

DEFINITION 7. *Let $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$ be two PFNs, then*

(1) $p_1 \ominus p_2 = \left(\sqrt{\frac{\mu_1^2-\mu_2^2}{1-\mu_2^2}}, \frac{\nu_1}{\nu_2}\right)$, *if* $\mu_1 \geq \mu_2, \nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$;
(2) $p_1 \oslash p_2 = \left(\frac{\mu_1}{\mu_2}, \sqrt{\frac{\nu_1^2-\nu_2^2}{1-\nu_2^2}}\right)$, *if* $\mu_1 \leq \min\{\mu_2, \frac{\mu_2\pi_1}{\pi_2}\}, \nu_1 \geq \nu_2$.

THEOREM 2. *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then*

(1) $(p^c)^\lambda = (\lambda p)^c$;
(2) $\lambda(p^c) = (p^\lambda)^c$.
(3) $p_1 \cup p_2 = p_2 \cup p_1$;
(4) $p_1 \cap p_2 = p_2 \cap p_1$;
(5) $\lambda(p_1 \cup p_2) = \lambda p_1 \cup \lambda p_2$;
(6) $(p_1 \cup p_2)^\lambda = p_1^\lambda \cup p_2^\lambda$;
(7) $\lambda(p_1 \ominus p_2) = \lambda p_1 \ominus \lambda p_2$, *if* $\mu_1 \geq \mu_2, \nu_1 \leq \min\{\nu_2, \frac{\nu_2 \pi_1}{\pi_2}\}$;
(8) $(p_1 \oslash p_2)^\lambda = p_1^\lambda \oslash p_2^\lambda$, *if* $\mu_1 \leq \min\{\mu_2, \frac{\mu_2 \pi_1}{\pi_2}\}, \nu_1 \geq \nu_2$;
(9) $\lambda_1 p \ominus \lambda_2 p = (\lambda_1 - \lambda_2)p$, *if* $\lambda_1 \geq \lambda_2$;
(10) $p^{\lambda_1} \oslash p^{\lambda_2} = p^{(\lambda_1 - \lambda_2)}$, *if* $\lambda_1 \geq \lambda_2$.

Proof. In the following, we shall prove (1), (3), (5), (7), (9) and (2), (4), (6), (8), (10) can be proved analogously.

$$(1)\ (p^c)^\lambda = (\nu, \mu)^\lambda = \left(\nu^\lambda, \sqrt{1-(1-\mu^2)^\lambda}\right),$$
$$(p^\lambda)^c = \left(\sqrt{1-(1-\mu^2)^\lambda}, \nu^\lambda\right)^c = \left(\nu^\lambda, \sqrt{1-(1-\mu^2)^\lambda}\right) = (p^c)^\lambda.$$

$$(3)\ p_1 \cup p_2 = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) = (\max\{\mu_2, \mu_1\}, \min\{\nu_2, \nu_1\}) = p_2 \cup p_1.$$

$$\begin{aligned}
(5)\ \lambda(p_1 \cup p_2) &= \lambda(\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \\
&= \left(\sqrt{1-\left(1-\max\left\{\mu_1^2, \mu_2^2\right\}\right)^\lambda}, \min\left\{\nu_1^\lambda, \nu_2^\lambda\right\}\right) \\
\lambda p_1 \cup \lambda p_2 &= \left(\sqrt{1-\left(1-\mu_1^2\right)^\lambda}, \nu_1^\lambda\right) \cup \left(\sqrt{1-\left(1-\mu_2^2\right)^\lambda}, \nu_2^\lambda\right) \\
&= \left(\max\left\{\sqrt{1-\left(1-\mu_1^2\right)^\lambda}, \sqrt{1-\left(1-\mu_1^2\right)^\lambda}\right\}, \min\left\{\nu_1^\lambda, \nu_2^\lambda\right\}\right) \\
&= \left(\sqrt{1-\left(1-\max\left\{\mu_1^2, \mu_2^2\right\}\right)^\lambda}, \min\left\{\nu_1^\lambda, \nu_2^\lambda\right\}\right) \\
&= \lambda(p_1 \cup p_2).
\end{aligned}$$

(7) Since $\mu_1 \geq \mu_2, \nu_1 \leq \min\{\nu_2, \frac{\nu_2 \pi_1}{\pi_2}\}$, we have

$$\begin{aligned}
&\nu_1 \pi_2 \leq \nu_2 \pi_1 \\
&\quad \Rightarrow \nu_1^2 \nu_2^2 + \nu_1^2 \pi_2^2 \leq \nu_1^2 \nu_2^2 + \pi_1^2 \nu_2^2 \\
&\quad \Rightarrow \frac{\nu_1^2}{\nu_2^2} \leq \frac{\nu_1^2 + \pi_1^2}{\nu_2^2 + \pi_2^2} \\
&\quad \Rightarrow \left(\frac{\nu_1^2}{\nu_2^2}\right)^\lambda \leq \left(\frac{\nu_1^2 + \pi_1^2}{\nu_2^2 + \pi_2^2}\right)^\lambda \\
&\quad \Rightarrow 1 - \left(\frac{\nu_1^2 + \pi_1^2}{\nu_2^2 + \pi_2^2}\right)^\lambda + \left(\frac{\nu_1^2}{\nu_2^2}\right)^\lambda \leq 1
\end{aligned}$$

$$\Rightarrow \left(\sqrt{1-\left(\frac{\nu_1^2+\pi_1^2}{\nu_2^2+\pi_2^2}\right)^{\lambda}}\right)^2+\left(\frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right)^2 \leq 1$$

$$\Rightarrow \left(\sqrt{1-\left(\frac{1-\mu_1^2}{1-\mu_2^2}\right)^{\lambda}}\right)^2+\left(\frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right)^2 \leq 1.\text{Then,}$$

$$\lambda(p_1 \ominus p_2) = \lambda\left(\sqrt{\frac{\mu_1^2-\mu_2^2}{1-\mu_2^2}}, \frac{\nu_1}{\nu_2}\right) = \left(\sqrt{1-\left(1-\frac{\mu_1^2-\mu_2^2}{1-\mu_2^2}\right)^{\lambda}}, \frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right)$$
$$= \left(\sqrt{1-\left(\frac{1-\mu_1^2}{1-\mu_2^2}\right)^{\lambda}}, \frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right),$$
$$\lambda p_1 \ominus \lambda p_2 = \left(\sqrt{1-\left(1-\mu_1^2\right)^{\lambda}}, \nu_1^{\lambda}\right) \ominus \left(\sqrt{1-\left(1-\mu_2^2\right)^{\lambda}}, \nu_2^{\lambda}\right)$$
$$= \left(\sqrt{\frac{1-\left(1-\mu_1^2\right)^{\lambda}-\left(1-\left(1-\mu_2^2\right)^{\lambda}\right)}{1-\left(1-\left(1-\mu_2^2\right)^{\lambda}\right)}}, \frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right)$$
$$= \left(\sqrt{1-\left(\frac{1-\mu_1^2}{1-\mu_2^2}\right)^{\lambda}}, \frac{\nu_1^{\lambda}}{\nu_2^{\lambda}}\right)$$
$$= \lambda(p_1 \ominus p_2).$$

(9) Since $\lambda_1 \geq \lambda_2$, then

$$\lambda_1 p \ominus \lambda_2 p = (\sqrt{1-(1-\mu^2)^{\lambda_1}}, \nu^{\lambda_1}) \ominus (\sqrt{1-(1-\mu^2)^{\lambda_2}}, \nu^{\lambda_2})$$
$$= \left(\sqrt{\frac{1-(1-\mu^2)^{\lambda_1}-(1-(1-\mu^2)^{\lambda_2})}{1-(1-(1-\mu^2)^{\lambda_2})}}, \frac{\nu^{\lambda_1}}{\nu^{\lambda_2}}\right)$$
$$= \left(\sqrt{1-(1-\mu^2)^{(\lambda_1-\lambda_2)}}, \nu^{(\lambda_1-\lambda_2)}\right)$$
$$= (\lambda_1-\lambda_2)p.$$

□

THEOREM 3. *Let* $p_1 = (\mu_1, \nu_1)$ *and* $p_2 = (\mu_2, \nu_2)$ *be two PFNs, then*

(1) $p_1^c \cup p_2^c = (p_1 \cap p_2)^c$; (2) $p_1^c \cap p_2^c = (p_1 \cup p_2)^c$;
(3) $p_1^c \oplus p_2^c = (p_1 \otimes p_2)^c$; (4) $p_1^c \otimes p_2^c = (p_1 \oplus p_2)^c$;
(5) $p_1^c \ominus p_2^c = (p_1 \oslash p_2)^c$, *if* $\nu_1 \geq \nu_2, \mu_1 \leq \min\{\mu_2, \frac{\mu_2\pi_1}{\pi_2}\}$;
(6) $p_1^c \oslash p_2^c = (p_1 \ominus p_2)^c$, *if* $\mu_1 \geq \mu_2, \nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$.

Proof. In the following, we shall prove (1), (3), (5) and (2), (4), (6) can be proved analogously.

$$(1)\ p_1^c \cup p_2^c = (\nu_1, \mu_1) \cup (\nu_2, \mu_2) = (\max\{\nu_1, \nu_2\}, \min\{\mu_1, \mu_2\})$$

$$(p_1 \cap p_2)^c = (\min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\})^c = (\max\{\nu_1, \nu_2\}, \min\{\mu_1, \mu_2\}) = p_1^c \cup p_2^c.$$

$$(3)\ p_1^c \oplus p_2^c = (\nu_1, \mu_1) \oplus (\nu_2, \mu_2) = (\sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2}, \mu_1\mu_2)$$

$$(p_1 \otimes p_2)^c = (\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2})^c = (\sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2}, \mu_1\mu_2) = p_1^c \oplus p_2^c.$$

(5) Since $\nu_1 \geq \nu_2$, $\mu_1 \leq \min\{\mu_2, \frac{\mu_2\pi_1}{\pi_2}\}$, we have

$$p_1^c \ominus p_2^c = (\nu_1, \mu_1) \ominus (\nu_2, \mu_2) = \left(\sqrt{\frac{\nu_1^2 - \nu_2^2}{1 - \nu_2^2}}, \frac{\mu_1}{\mu_2}\right)$$

$$(p_1 \oslash p_2)^c = \left(\frac{\mu_1}{\mu_2}, \sqrt{\frac{\nu_1^2 - \nu_2^2}{1 - \nu_2^2}}\right)^c = \left(\sqrt{\frac{\nu_1^2 - \nu_2^2}{1 - \nu_2^2}}, \frac{\mu_1}{\mu_2}\right) = p_1^c \ominus p_2^c.$$

□

THEOREM 4. *Let* $p_1 = (\mu_1, \nu_1)$ *and* $p_2 = (\mu_2, \nu_2)$ *be two PFNs, then*

(1) $(p_1 \cup p_2) \oplus (p_1 \cap p_2) = p_1 \oplus p_2$;
(2) $(p_1 \cup p_2) \otimes (p_1 \cap p_2) = p_1 \otimes p_2$;
(3) $(p_1 \cup p_2) \ominus (p_1 \cap p_2) = p_1 \ominus p_2$, *if* $\mu_1 \geq \mu_2$, $\nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$;
(4) $(p_1 \cup p_2) \oslash (p_1 \cap p_2) = p_1 \oslash p_2$, *if* $\nu_1 \geq \nu_2$, $\mu_1 \leq \min\{\mu_2, \frac{\mu_2\pi_1}{\pi_2}\}$.

Proof. In the following, we shall prove (1), (3) and (2), (4) can be proved analogously.

$$
\begin{aligned}
(1)\ & (p_1 \cup p_2) \oplus (p_1 \cap p_2) \\
&= (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \oplus (\min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\}) \\
&= \Bigg(\sqrt{\max\left\{\mu_1^2, \mu_2^2\right\} + \min\left\{\mu_1^2, \mu_2^2\right\} - \max\left\{\mu_1^2, \mu_2^2\right\}\min\left\{\mu_1^2, \mu_2^2\right\}}, \\
&\qquad \min\{\nu_1, \nu_2\}\max\{\nu_1, \nu_2\}\Bigg) \\
&= \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2\right) \\
&= p_1 \oplus p_2.
\end{aligned}
$$

(3) Since $\mu_1 \geq \mu_2$, $\nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$, then

$$\begin{aligned}(p_1 \cup p_2) \ominus (p_1 \cap p_2) &= (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \ominus (\min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\}) \\ &= \left(\sqrt{\frac{\max\left\{\mu_1^2, \mu_2^2\right\} - \min\left\{\mu_1^2, \mu_2^2\right\}}{1 - \min\left\{\mu_1^2, \mu_2^2\right\}}}, \frac{\min\{\nu_1, \nu_2\}}{\max\{\nu_1, \nu_2\}}\right) \\ &= \left(\sqrt{\frac{\mu_1^2 - \mu_2^2}{1 - \mu_2^2}}, \frac{\nu_1}{\nu_2}\right) \\ &= p_1 \ominus p_2.\end{aligned}$$

□

THEOREM 5. *Let $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$ be two PFNs, then*

(1) $(p_1 \cup p_2) \cap p_2 = p_2$;
(2) $(p_1 \cap p_2) \cup p_2 = p_2$;
(3) $(p_1 \ominus p_2) \oplus p_2 = p_1$, *if* $\mu_1 \geq \mu_2$, $\nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$;
(4) $(p_1 \oslash p_2) \otimes p_2 = p_1$, *if* $\nu_1 \geq \nu_2$, $\mu_1 \leq \min\{\mu_2, \frac{\mu_2\pi_1}{\pi_2}\}$.

Proof. In the following, we shall prove (1), (3) and (2), (4) can be proved analogously.

(1) $(p_1 \cup p_2) \cap p_2 = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \cap (\mu_2, \nu_2)$

$$\begin{aligned}&= (\min\{\max\{\mu_1, \mu_2\}, \mu_2\}, \max\{\min\{\nu_1, \nu_2\}, \nu_2\}) \\ &= (\mu_2, \nu_2) = p_2.\end{aligned}$$

(3) Since $\mu_1 \geq \mu_2$, $\nu_1 \leq \min\{\nu_2, \frac{\nu_2\pi_1}{\pi_2}\}$, then

$$\begin{aligned}(p_1 \ominus p_2) \oplus p_2 &= \left(\sqrt{\frac{\mu_1^2 - \mu_2^2}{1 - \mu_2^2}}, \frac{\nu_1}{\nu_2}\right) \oplus (\mu_2, \nu_2) \\ &= \left(\sqrt{\left(\sqrt{\frac{\mu_1^2 - \mu_2^2}{1 - \mu_2^2}}\right)^2 + \mu_2^2 - \left(\sqrt{\frac{\mu_1^2 - \mu_2^2}{1 - \mu_2^2}}\right)^2 \mu_2^2}, \quad \frac{\nu_1}{\nu_2}\nu_2\right) \\ &= (\mu_1, \nu_1) = p_1.\end{aligned}$$

□

THEOREM 6. *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, then*

(1) $(p_1 \cup p_2) \cap p_3 = (p_1 \cap p_3) \cup (p_2 \cap p_3)$;
(2) $(p_1 \cap p_2) \cup p_3 = (p_1 \cup p_3) \cap (p_2 \cup p_3)$;
(3) $(p_1 \cup p_2) \oplus p_3 = (p_1 \oplus p_3) \cup (p_2 \oplus p_3)$;
(4) $(p_1 \cap p_2) \oplus p_3 = (p_1 \oplus p_3) \cap (p_2 \oplus p_3)$;
(5) $(p_1 \cup p_2) \otimes p_3 = (p_1 \otimes p_3) \cup (p_2 \otimes p_3)$;
(6) $(p_1 \cap p_2) \otimes p_3 = (p_1 \otimes p_3) \cap (p_2 \otimes p_3)$;
(7) $(p_1 \cup p_2) \ominus p_3 = (p_1 \ominus p_3) \cup (p_2 \ominus p_3)$, *if* $\mu_3 \leq \min\{\mu_1, \mu_2\}$, $\max\{\nu_1, \nu_2\} \leq \{\nu_3, \nu_3\frac{\pi_1}{\pi_3}, \nu_3\frac{\pi_2}{\pi_3}\}$, $\frac{\max\{\mu_1^2, \mu_2^2\} - \mu_3^2}{1 - \mu_3^2} + \frac{\min\{\nu_1^2, \nu_2^2\}}{\nu_3^2} \leq 1$;

(8) $(p_1 \cap p_2) \ominus p_3 = (p_1 \ominus p_3) \cap (p_2 \ominus p_3)$, *if* $\mu_3 \leq \min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\} \leq \{\nu_3, \nu_3 \frac{\pi_1}{\pi_3}, \nu_3 \frac{\pi_2}{\pi_3}\}, \frac{\min\{\mu_1^2, \mu_2^2\} - \mu_3^2}{1-\mu_3^2} + \frac{\max\{\nu_1^2, \nu_2^2\}}{\nu_3^2} \leq 1$;

(9) $(p_1 \cup p_2) \oslash p_3 = (p_1 \oslash p_3) \cup (p_2 \oslash p_3)$, *if* $\nu_3 \leq \min\{\nu_1, \nu_2\}, \max\{\mu_1, \mu_2\} \leq \{\mu_3, \mu_3 \frac{\pi_1}{\pi_3}, \mu_3 \frac{\pi_2}{\pi_3}\}, \frac{\min\{\nu_1^2, \nu_2^2\} - \nu_3^2}{1-\nu_3^2} + \frac{\max\{\mu_1^2, \mu_2^2\}}{\mu_3^2} \leq 1$;

(10) $(p_1 \cap p_2) \oslash p_3 = (p_1 \oslash p_3) \cap (p_2 \oslash p_3)$, *if* $\nu_3 \leq \min\{\nu_1, \nu_2\}, \max\{\mu_1, \mu_2\} \leq \{\mu_3, \mu_3 \frac{\pi_1}{\pi_3}, \mu_3 \frac{\pi_2}{\pi_3}\}, \frac{\max\{\nu_1^2, \nu_2^2\} - \nu_3^2}{1-\nu_3^2} + \frac{\min\{\mu_1^2, \mu_2^2\}}{\mu_3^2} \leq 1$.

Proof. In the following, we shall prove the (1), (3), (5), (7), (9) and the (2), (4), (6), (8), (10) can be proved analogously.

(1) $(p_1 \cup p_2) \cap p_3$

$$\begin{aligned}
&= (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \cap (\mu_3, \nu_3) \\
&= (\min\{\max\{\mu_1, \mu_2\}, \mu_3\}, \max\{\min\{\nu_1, \nu_2\}, \nu_3\}) \\
&= (\max\{\min\{\mu_1, \mu_3\}, \min\{\mu_2, \mu_3\}\}, \min\{\max\{\nu_1, \nu_3\}, \max\{\nu_2, \nu_3\}\}) \\
&= (\min\{\mu_1, \mu_3\}, \max\{\nu_1, \nu_3\}) \cup (\min\{\mu_2, \mu_3\}, \max\{\nu_2, \nu_3\}) \\
&= (p_1 \cap p_3) \cup (p_2 \cap p_3).
\end{aligned}$$

(3) $(p_1 \cup p_2) \oplus p_3$

$$\begin{aligned}
&= (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \oplus (\mu_3, \nu_3) \\
&= \left(\sqrt{\max\{\mu_1^2, \mu_2^2\} + \mu_3^2 - \max\{\mu_1^2, \mu_2^2\}\mu_3^2}, \min\{\nu_1, \nu_2\}\nu_3\right) \\
&= \left(\sqrt{(1-\mu_3^2)\max\{\mu_1^2, \mu_2^2\} + \mu_3^2}, \min\{\nu_1\nu_3, \nu_2\nu_3\}\right)
\end{aligned}$$

$(p_1 \oplus p_3) \cup (p_2 \oplus p_3)$

$$\begin{aligned}
&= \left(\sqrt{\mu_1^2 + \mu_3^2 - \mu_1^2\mu_3^3}, \nu_1\nu_3\right) \cup \left(\sqrt{\mu_2^2 + \mu_3^2 - \mu_2^2\mu_3^2}, \nu_2\nu_3\right) \\
&= \left(\max\{\sqrt{\mu_1^2 + \mu_3^2 - \mu_1^2\mu_3^2}, \sqrt{\mu_2^2 + \mu_3^2 - \mu_2^2\mu_3^2}\}, \min\{\nu_1\nu_3, \nu_2\nu_3\}\right) \\
&= \left(\max\{\sqrt{(1-\mu_3^2)\mu_1^2 + \mu_3^2}, \sqrt{(1-\mu_3^2)\mu_2^2 + \mu_3^2}\}, \min\{\nu_1\nu_3, \nu_2\nu_3\}\right) \\
&= \left(\sqrt{(1-\mu_3^2)\max\{\mu_1^2, \mu_2^2\} + \mu_3^2}, \min\{\nu_1\nu_3, \nu_2\nu_3\}\right) \\
&= (p_1 \cup p_2) \oplus p_3.
\end{aligned}$$

(5) $(p_1 \cup p_2) \otimes p_3$

$$\begin{aligned}
&= (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \otimes (\mu_3, \nu_3) \\
&= \left(\max\{\mu_1, \mu_2\}\mu_3, \sqrt{\min\{\nu_1^2, \nu_2^2\} + \nu_3^2 - \min\{\nu_1^2, \nu_2^2\}\nu_3^2}\right) \\
&= \left(\max\{\mu_1, \mu_2\}\mu_3, \sqrt{(1-\nu_3^2)\min\{\nu_1^2, \nu_2^2\} + \nu_3^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&(p_1 \otimes p_3) \cup (p_2 \otimes p_3) \\
&\quad = (\mu_1\mu_3, \sqrt{\nu_1^2 + \nu_3^2 - \nu_1^2\nu_3^2}) \cup (\mu_2\mu_3, \sqrt{\nu_2^2 + \nu_3^2 - \nu_2^2\nu_3^2}) \\
&\quad = \left(\max\{\mu_1\mu_3, \mu_2\mu_3\}, \min\left\{\sqrt{\nu_1^2 + \nu_3^2 - \nu_1^2\nu_3^2}, \sqrt{\nu_2^2 + \nu_3^2 - \nu_2^2\nu_3^2}\right\}\right) \\
&\quad = \left(\max\{\mu_1, \mu_2\}\mu_3, \sqrt{(1 - \nu_3^2)\min\{\nu_1^2, \nu_2^2\} + \nu_3^2}\right) \\
&\quad = (p_1 \cup p_2) \otimes p_3.
\end{aligned}
$$

(7) Since $\mu_3 \leq \min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\} \leq \{\nu_3, \nu_3\frac{\pi_1}{\pi_3}, \nu_3\frac{\pi_2}{\pi_3}\}, \frac{\max\{\mu_1^2,\mu_2^2\}-\mu_3^2}{1-\mu_3^2} + \frac{\min\{\nu_1^2,\nu_2^2\}}{\nu_3^2} \leq 1,$ then

$$
\begin{aligned}
&(p_1 \cup p_2) \ominus p_3 \\
&\quad = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \ominus (\mu_3, \nu_3) \\
&\quad = \left(\sqrt{\frac{\max\{\mu_1^2, \mu_2^2\} - \mu_3^2}{1 - \mu_3^2}}, \frac{\min\{\nu_1, \nu_2\}}{\nu_3}\right) \\
&\quad = \left(\max\left\{\sqrt{\frac{\mu_1^2 - \mu_3^2}{1 - \mu_3^2}}, \sqrt{\frac{\mu_2^2 - \mu_3^2}{1 - \mu_3^2}}\right\}, \min\{\frac{\nu_1}{\nu_3}, \frac{\nu_2}{\nu_3}\}\right) \\
&(p_1 \ominus p_3) \cup (p_2 \ominus p_3) \\
&\quad = \left(\sqrt{\frac{\mu_1^2 - \mu_3^2}{1 - \mu_3^2}}, \frac{\nu_1}{\nu_3}\right) \cup \left(\sqrt{\frac{\mu_2^2 - \mu_3^2}{1 - \mu_3^2}}, \frac{\nu_2}{\nu_3}\right) \\
&\quad = \left(\max\left\{\sqrt{\frac{\mu_1^2 - \mu_3^2}{1 - \mu_3^2}}, \sqrt{\frac{\mu_2^2 - \mu_3^2}{1 - \mu_3^2}}\right\}, \min\left\{\frac{\nu_1}{\nu_3}, \frac{\nu_2}{\nu_3}\right\}\right) \\
&\quad = (p_1 \cup p_2) \ominus p_3.
\end{aligned}
$$

(9) Since $\nu_3 \leq \min\{\nu_1, \nu_2\}, \max\{\mu_1, \mu_2\} \leq \{\mu_3, \mu_3\frac{\pi_1}{\pi_3}, \mu_3\frac{\pi_2}{\pi_3}\}, \frac{\min\{\nu_1^2,\nu_2^2\}-\nu_3^2}{1-\nu_3^2} + \frac{\max\{\mu_1^2,\mu_2^2\}}{\mu_3^2} \leq 1,$ then

$$
\begin{aligned}
&(p_1 \cup p_2) \oslash p_3 \\
&\quad = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}) \oslash (\mu_3, \nu_3) \\
&\quad = \left(\frac{\max(\mu_1, \mu_2)}{\mu_3}, \sqrt{\frac{\min\{\nu_1^2, \nu_2^2\} - \nu_3^2}{1 - \nu_3^2}}\right) \\
&\quad = \left(\max\left\{\frac{\mu_1}{\mu_3}, \frac{\mu_2}{\mu_3}\right\}, \min\left\{\sqrt{\frac{\nu_1^2 - \nu_3^2}{1 - \nu_3^2}}, \sqrt{\frac{\nu_2^2 - \nu_3^2}{1 - \nu_3^2}}\right\}\right) \\
&(p_1 \oslash p_3) \cup (p_2 \oslash p_3)
\end{aligned}
$$

$$= \left(\frac{\mu_1}{\mu_3}, \sqrt{\frac{\nu_1^2 - \nu_3^2}{1 - \nu_3^2}}\right) \cup \left(\frac{\mu_2}{\mu_3}, \sqrt{\frac{\nu_2^2 - \nu_3^2}{1 - \nu_3^2}}\right)$$

$$= \left(\max\left\{\frac{\mu_1}{\mu_3}, \frac{\mu_2}{\mu_3}\right\}, \min\left\{\sqrt{\frac{\nu_1^2 - \nu_3^2}{1 - \nu_3^2}}, \sqrt{\frac{\nu_2^2 - \nu_3^2}{1 - \nu_3^2}}\right\}\right)$$

$$= (p_1 \cup p_2) \oslash p_3.$$

□

THEOREM 7. *Let* $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, *and* $p_2 = (\mu_2, \nu_2)$ *be three PFNs, then*

(1) $p_1 \cup p_2 \cup p_3 = p_1 \cup p_3 \cup p_2$;
(2) $p_1 \cap p_2 \cap p_3 = p_1 \cap p_3 \cap p_2$;
(3) $p_1 \oplus p_2 \oplus p_3 = p_1 \oplus p_3 \oplus p_2$;
(4) $p_1 \otimes p_2 \otimes p_3 = p_1 \otimes p_3 \otimes p_2$;
(5) $p_1 \ominus p_2 \ominus p_3 = p_1 \ominus p_3 \ominus p_2$, *if* $\mu_1 \geq \max\{\mu_2, \mu_3\}$, $\nu_1 \leq \min\{\nu_2\nu_3, \nu_2\frac{\pi_1}{\pi_2}, \nu_3\frac{\pi_1}{\pi_3}\}$, $\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{(1-\mu_2^2)(1-\mu_3^2)} + \frac{\nu_1^2}{\nu_2^2\nu_3^2} \leq 1$;
(6) $p_1 \oslash p_2 \oslash p_3 = p_1 \oslash p_3 \oslash p_2$, *if* $\nu_1 \geq \max\{\nu_2, \nu_3\}$, $\mu_1 \leq \min\{\mu_2\mu_3, \mu_2\frac{\pi_1}{\pi_2}, \mu_3\frac{\pi_1}{\pi_3}\}$, $\frac{\nu_1^2-\nu_2^2-\nu_3^2+\nu_2^2\nu_3^2}{(1-\nu_2^2)(1-\nu_3^2)} + \frac{\mu_1^2}{\mu_2^2\mu_3^2} \leq 1$.

Proof. In the following, we shall prove the (1), (3), (5) and the (2), (4), (6) can be proved analogously.

(1) $p_1 \cup p_2 \cup p_3$

$$= (\mu_1, \nu_1) \cup (\mu_2, \nu_2) \cup (\mu_3, \nu_3)$$
$$= (\max\{\max\{\mu_1, \mu_2\}, \mu_3\}, \min\{\min\{\nu_1, \nu_2\}, \nu_3\})$$
$$= (\max\{\max\{\mu_1, \mu_3\}, \mu_2\}, \min\{\min\{\nu_1, \nu_3\}, \nu_2\})$$
$$= p_1 \cup p_3 \cup p_2.$$

(3) $p_1 \oplus p_2 \oplus p_3$

$$= (\mu_1, \nu_1) \oplus (\mu_2, \nu_2) \oplus (\mu_3, \nu_3)$$
$$= \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2\right) \oplus (\mu_3, \nu_3)$$
$$= \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2 + \mu_3^2 - \left(\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2\right)\mu_3^2}, \nu_1\nu_3\nu_3\right)$$
$$= \left(\sqrt{\mu_1^2 + \mu_3^2 - \mu_1^2\mu_2^2 + \mu_2^2 - \left(\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2\right)\mu_3^2}, \nu_1\nu_3\nu_3\right)$$
$$= p_1 \oplus p_3 \oplus p_2.$$

(5) Since $\nu_1 \leq \min\{\nu_2\nu_3, \nu_2\frac{\pi_1}{\pi_2}, \nu_3\frac{\pi_1}{\pi_3}\}$, $\mu_1 \geq \max\{\mu_2, \mu_3\}$, $\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{(1-\mu_2^2)(1-\mu_3^2)} + \frac{\nu_1^2}{\nu_2^2\nu_3^2} \leq 1$, then

$p_1 \ominus p_2 \ominus p_3$

$$= \left(\sqrt{\frac{\mu_1^2-\mu_2^2}{1-\mu_2^2}}, \frac{\nu_1}{\nu_2}\right) \ominus (\mu_3, \nu_3) = \left(\sqrt{\frac{\frac{\mu_1^2-\mu_2^2}{1-\mu_2^2}-\mu_3^2}{1-\mu_3^2}}, \frac{\nu_1}{\nu_2\nu_3}\right)$$

$$= \left(\sqrt{\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{\left(1-\mu_2^2\right)\left(1-\mu_3^2\right)}}, \frac{\nu_1}{\nu_2\nu_3}\right)$$

$p_1 \ominus p_3 \ominus p_2$

$$= \left(\sqrt{\frac{\mu_1^2-\mu_3^2}{1-\mu_3^2}}, \frac{\nu_1}{\nu_3}\right) \ominus (\mu_2, \nu_2) = \left(\sqrt{\frac{\frac{\mu_1^2-\mu_3^2}{1-\mu_3^2}-\mu_2^2}{1-\mu_2^2}}, \frac{\nu_1}{\nu_2\nu_3}\right)$$

$$= \left(\sqrt{\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{\left(1-\mu_2^2\right)\left(1-\mu_3^2\right)}}, \frac{\nu_1}{\nu_2\nu_3}\right)$$

$$= p_1 \ominus p_2 \ominus p_3.$$

□

THEOREM 8. *Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PFNs, then*

(1) $p_1 \ominus p_2 \ominus p_3 = p_1 \ominus (p_2 \oplus p_3)$, *if* $\mu_1 \geq \max\{\mu_2, \mu_3\}$, $\nu_1 \leq \min\{\nu_2\nu_3, \frac{\nu_2\pi_1}{\pi_2}\}$, $\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{(1-\mu_2^2)(1-\mu_3^2)} + \frac{\nu_1^2}{\nu_2^2\nu_3^2} \leq 1$;

(2) $p_1 \oslash p_2 \oslash p_3 = p_1 \oslash (p_2 \otimes p_3)$, *if* $\nu_1 \geq \max\{\nu_2, \nu_3\}$, $\mu_1 \leq \min\{\mu_2\mu_3, \frac{\mu_2\pi_1}{\pi_2}\}$, $\frac{\nu_1^2-\nu_2^2-\nu_3^2+\nu_2^2\nu_3^2}{(1-\nu_2^2)(1-\nu_3^2)} + \frac{\mu_1^2}{\mu_2^2\mu_3^2} \leq 1$.

Proof. For three PFNs p, p_1, p_2, we have

(1) Since $\mu_1 \geq \mu_2$, $\nu_1 \leq \min\{\nu_2\nu_3, \frac{\nu_2\pi_1}{\pi_2}\}$, $\frac{\nu_1^2}{\nu_2^2\nu_3^2} + \frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{(1-\mu_2^2)(1-\mu_3^2)} \leq 1$, then

$$p_1 \ominus p_2 \ominus p_3 = (\mu_1, \nu_1) \ominus (\mu_2, \nu_2) \ominus (\mu_3, \nu_3)$$
$$= \left(\sqrt{\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{\left(1-\mu_2^2\right)\left(1-\mu_3^2\right)}}, \frac{\nu_1}{\nu_2\nu_3}\right)$$
$$p_1 \ominus (p_2 \oplus p_3) = (\mu_1, \nu_1) \ominus \left(\sqrt{\mu_2^2+\mu_3^2-\mu_2^2\mu_3^2}, \nu_2\nu_3\right)$$
$$= \left(\sqrt{\frac{\mu_1^2-\mu_2^2-\mu_3^2+\mu_2^2\mu_3^2}{\left(1-\mu_2^2\right)\left(1-\mu_3^2\right)}}, \frac{\nu_1}{\nu_2\nu_3}]\right)$$
$$= p_1 \ominus p_2 \ominus p_3$$

(2) Since $\nu_1 \geq \nu_2$, $\mu_1 \leq \min\{\mu_2\mu_3, \frac{\mu_2\pi_1}{\pi_2}\}$, $\frac{\mu_1^2}{\mu_2^2\mu_3^2} + \frac{\nu_1^2-\nu_2^2-\nu_3^2+\nu_2^2\nu_3^2}{(1-\nu_2^2)(1-\nu_3^2)} \leq 1$, then

$$p_1 \oslash p_2 \oslash p_3 = (\mu_1, \nu_1) \oslash (\mu_2, \nu_2) \oslash (\mu_3, \nu_3)$$
$$= \left(\frac{\mu_1}{\mu_2\mu_3}, \sqrt{\frac{\nu_1^2-\nu_2^2-\nu_3^2+\nu_2^2\nu_3^2}{\left(1-\nu_2^2\right)\left(1-\nu_3^2\right)}}\right)$$

$$
\begin{aligned}
p_1 \oslash (p_2 \otimes p_3) &= (\mu_1, \nu_1) \oslash \left(\mu_2\mu_3, \sqrt{\nu_2^2 + \nu_3^2 - \nu_2^2\nu_3^2}\right) \\
&= \left(\frac{\mu_1}{\mu_2\mu_3}, \sqrt{\frac{\nu_1^2 - \nu_2^2 - \nu_3^2 + \nu_2^2\nu_3^2}{\left(1-\nu_2^2\right)\left(1-\nu_3^2\right)}}\right) \\
&= p_1 \oslash p_2 \oslash p_3
\end{aligned}
$$

□

4. PYTHAGOREAN FUZZY AGGREGATION OPERATORS AND THEIR PROPERTIES

In this section, we introduce the Pythagorean fuzzy aggregation operators proposed by Yager.[9,10] Then, some desirable properties, such as boundedness, idempotency, and monotonicity, are discussed in detail.

DEFINITION 8 ([9]). *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p_i(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} = 1$, then a PFWA operator is a mapping PFWA: $P^n \rightarrow P$, where*

$$
PFWA(p_1, p_2, \ldots, p_n) = \left(\sum_{i=1}^{n} w_i\mu_i, \sum_{i=1}^{n} w_i\nu_i\right). \tag{6}
$$

DEFINITION 9 ([10]). *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p_i(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} = 1$, then a PFWG operator is a mapping PFWG: $P^n \rightarrow P$, where*

$$
PFWG(p_1, p_2, \ldots, p_n) = \left(\prod_{i=1}^{n} \mu_i^{w_i}, \prod_{i=1}^{n} \nu_i^{w_i}\right). \tag{7}
$$

DEFINITION 10 ([10]). *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p_i(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} = 1$, then a PFWPA operator is a mapping PFWPA: $P^n \rightarrow P$, where*

$$
PFWPA(p_1, p_2, \ldots, p_n) = \left(\left(\sum_{i=1}^{n} w_i\mu_i^2\right)^{1/2}, \left(\sum_{i=1}^{n} w_i\nu_i^2\right)^{1/2}\right). \tag{8}
$$

DEFINITION 11 ([10]). *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p_i(i = 1, 2, \ldots, n)$, with $\sum_{i=1}^{n} = 1$,* *then a PFWPG operator is a mapping PFWPG:* $P^n \rightarrow P$*, where*

$$PFWPG(p_1, p_2, \ldots, p_n) = \left(\left(1 - \prod_{i=1}^{n}\left(1-\mu_i^2\right)^{w_i}\right)^{1/2}, \left(1 - \prod_{i=1}^{n}\left(1-\nu_i^2\right)^{w_i}\right)^{1/2}\right). \tag{9}$$

LEMMA 1 ([19]). *Let* $x_i > 0$, $w_i > 0$, $i = 1, 2, \ldots, n$, *and* $\sum_{i=1}^{n} w_i = 1$, *then*

$$\prod_{i=1}^{n}(x_i)^{w_i} \leq \sum_{i=1}^{n} w_i x_i \tag{10}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

THEOREM 9. *Let* $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ *be a collection of PFNs,* $p = (\mu, \nu)$ *is also PFN, and* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of them,* $\sum_{i=1}^{n} w_i = 1$, *then*

(1) *PFWA* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ *PFWA* $(p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p)$;
(2) *PFWPA* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ *PFWPA* $(p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p)$;
(3) *PFWPG* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ *PFWPG* $(p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p)$;
(4) *PFWG* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ *PFWG* $(p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p)$.

Proof. In the following, we shall prove the (1), (3) and the (2), (4) can be proved analogously.

(1) For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we can get

$$\sqrt{\mu_i^2 + \mu^2 - \mu_i^2\mu^2} \geq \sqrt{2\mu_i^2\mu^2 - \mu_i^2\mu^2} = \mu_i\mu,$$
$$\sqrt{\nu_i^2 + \nu^2 - \nu_i^2\nu^2} \geq \sqrt{2\nu_i^2\nu^2 - \nu_i^2\nu^2} = \nu_i\nu,$$

i.e.,

$$\sum_{i=1}^{n} w_i\sqrt{\mu_i^2 + \mu^2 - \mu_i^2\mu^2} \geq \sum_{i=1}^{n} w_i\mu_i\mu, \sum_{i=1}^{n} w_i\sqrt{\nu_i^2 + \nu^2 - \nu_i^2\nu^2} \geq \sum_{i=1}^{n} w_i\nu_i\nu.$$

Since *PFWA* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) = (\sum_{i=1}^{n} w_i\sqrt{\mu_i^2 + \mu^2 - \mu_i^2\mu^2}, \sum_{i=1}^{n} w_i\nu_i\nu)$,

$$PFWA\ (p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p) = \left(\sum_{i=1}^{n} w_i\mu_i\mu, \sum_{i=1}^{n} w_i\sqrt{\nu_i^2 + \nu^2 - \nu_i^2\nu^2}\right).$$

According to Definition 3, the proof is proved.

(3) For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we can get

$$\begin{aligned}
&\mu_i^2 + \mu^2 - \mu_i^2\mu^2 \geq 2\mu_i^2\mu^2 - \mu_i^2\mu^2 = \mu_i^2\mu^2 \\
&\quad \Rightarrow 1 - \left(\mu_i^2 + \mu^2 - \mu_i^2\mu^2\right) \leq 1 - \mu_i^2\mu^2 \\
&\quad \Rightarrow \left(1 - \left(\mu_i^2 + \mu^2 - \mu_i^2\mu^2\right)\right)^{w_i} \leq \left(1 - \mu_i^2\mu^2\right)^{w_i} \\
&\quad \Rightarrow \prod_{i=1}^{n}\left(1 - \left(\mu_i^2 + \mu^2 - \mu_i^2\mu^2\right)\right)^{w_i} \leq \prod_{i=1}^{n}\left(1 - \mu_i^2\mu^2\right)^{w_i} \\
&\quad \Rightarrow 1 - \prod_{i=1}^{n}\left(1 - \left(\mu_i^2 + \mu^2 - \mu_i^2\mu^2\right)\right)^{w_i} \geq 1 - \prod_{i=1}^{n}\left(1 - \mu_i^2\mu^2\right)^{w_i}
\end{aligned}$$

Similarly,

$$1 - \prod_{i=1}^{n}\left(1 - \left(\nu_i^2 + \nu^2 - \nu_i^2\nu^2\right)\right)^{w_i} \geq 1 - \prod_{i=1}^{n}\left(1 - \nu_i^2\nu^2\right)^{w_i}$$

Since $PFWPG\,(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p)$

$$= \left(\left(1 - \prod_{i=1}^{n}\left(1 - \left(\mu_i^2 + \mu^2 - \mu_i^2\mu^2\right)\right)^{w_i}\right)^{1/2}, \left(1 - \prod_{i=1}^{n}(1 - \nu_i^2\nu^2)^{w_i}\right)^{1/2}\right),$$

$PFWPG\,(p_1 \otimes p, p_2 \otimes p, \ldots, p_n \otimes p)$

$$= \left(\left(1 - \prod_{i=1}^{n}\left(1 - \mu_i^2\mu^2\right)^{w_i}\right)^{1/2}, \left(1 - \prod_{i=1}^{n}\left(1 - \left(\nu_i^2 + \nu^2 - \nu_i^2\nu^2\right)\right)^{w_i}\right)^{1/2}\right).$$

According to Definition 3, the proof is proved. □

THEOREM 10. *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs, $p = (\mu, \nu)$ is also PFN, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} w_i = 1$, $\mu_i \geq \sqrt{\frac{\mu^2 + \sqrt{4\mu^2 - 3\mu^4}}{2}}$, $\nu_i \leq \min\{\nu, \nu\frac{\pi_i}{\pi}\}$, then*

(1) $PFWA\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) \geq PFWA\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n)$;
(2) $PFWPA\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) \geq PFWPA\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n)$;
(3) $PFWG\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) \geq PFWG\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n)$;
(4) $PFWPG\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) \geq PFWPG\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n)$.

Proof. In the following, we shall prove the (1)(3) and the (2)(4) can be proved analogously.

(1) By Definitions 7 and 8, we have that

$$PFWA\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) = \left(\sum_{i=1}^{n} w_i\sqrt{\frac{\mu_i^2 - \mu^2}{1 - \mu^2}}, \sum_{i=1}^{n} w_i\frac{\nu_i}{\nu}\right),$$

$$PFWA\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n) = \left(\sum_{i=1}^{n} w_i\frac{\mu}{\mu_i}, \sum_{i=1}^{n} w_i\sqrt{\frac{\nu^2 - \nu_i^2}{1 - \nu_i^2}}\right).$$

For a given $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, in the following, we denote $f(\mu_i) = \frac{\mu_i^2-\mu^2}{1-\mu^2} - \frac{\mu^2}{\mu_i^2} = \frac{\mu_i^4-\mu^2\mu_i^2-\mu^2+\mu^4}{\mu_i^2(1-\mu^2)}$ and prove that $f(\mu_i) \geq 0$, because $\mu_i^2(1-\mu^2) \geq 0$, so we only prove $g(\mu_i) = \mu_i^4 - \mu^2\mu_i^2 - \mu^2 + \mu^4 \geq 0$.
We take the derivative of $g(\mu_i)$ and obtain $g'(\mu_i) = 4\mu_i^3 - 2\mu^2\mu_i = \mu_i(4\mu_i^2 - 2\mu^2)$, because $\mu^2 \geq \mu^4$, $\mu_i \geq \sqrt{\frac{\mu^2+\sqrt{4\mu^2-3\mu^4}}{2}} \geq \sqrt{\frac{\mu^2+\sqrt{4\mu^4-3\mu^4}}{2}} = \mu$, therefore, $g'(\mu_i) \geq 0$, i.e., it is monotonically increasing, $g(\mu_i)_{\min} = g(\sqrt{\frac{\mu^2+\sqrt{4\mu^2-3\mu^4}}{2}}) = 0$, i.e., $f(\mu_i) \geq 0$.
Hence, $\frac{\mu_i^2-\mu^2}{1-\mu^2} \geq \frac{\mu^2}{\mu_i^2} \Rightarrow \sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}} \geq \frac{\mu}{\mu_i} \Rightarrow \sum_{i=1}^n w_i\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}} \geq \sum_{i=1}^n w_i \frac{\mu}{\mu_i}$.
Similarly,

$$\sqrt{\frac{\nu^2-\nu_i^2}{1-\nu_i^2}} \geq \frac{\nu_i}{\nu} \Rightarrow \sum_{i=1}^n w_i\sqrt{\frac{\nu^2-\nu_i^2}{1-\nu_i^2}} \geq \sum_{i=1}^n w_i \frac{\nu_i}{\nu}.$$

Later, we prove the constraint condition, $\nu_i\pi \leq \nu\pi_i \Rightarrow \nu_i^2\pi^2 \leq \nu^2\pi_i^2 \Rightarrow \nu_i^2(\nu^2+\pi^2) \leq \nu^2(\nu_i^2+\pi_i^2) \Rightarrow \nu_i^2(1-\mu^2) \leq \nu^2(1-\mu_i^2) \Rightarrow (\mu_i^2-\mu^2)\nu^2 + \nu_i^2(1-\mu^2) \leq (1-\mu^2)\nu^2 \Rightarrow \frac{\mu_i^2-\mu^2}{1-\mu^2} + \frac{\nu_i^2}{\nu^2} \leq 1$. Similarly, $\frac{\nu^2-\nu_i^2}{1-\nu_i^2} + \frac{\mu^2}{\mu_i^2} \leq 1$.
Because, $(\sum_{i=1}^n w_i\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}})^2 + (\sum_{i=1}^n w_i \frac{\nu_i}{\nu})^2 \leq (\sum_{i=1}^n w_i\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}} + \sum_{i=1}^n w_i \frac{\nu_i}{\nu})^2 = \sum_{i=1}^n w_i(\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}} + \frac{\nu_i}{\nu}) \leq \sum_{i=1}^n w_i = 1$. Similarly, $(\sum_{i=1}^n w_i \frac{\mu}{\mu_i})^2 + (\sum_{i=1}^n w_i\sqrt{\frac{\nu^2-\nu_i^2}{1-\nu_i^2}})^2 \leq 1$.
Thus, according to Definition 3, the proof is proved.

(3) By Definitions 7 and 8, we have that

$$PFWG\,(p_1 \ominus p, p_2 \ominus p, \ldots, p_n \ominus p) = \left(\prod_{i=1}^n \left(\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}}\right)^{w_i}, \prod_{i=1}^n \left(\frac{\nu_i}{\nu}\right)^{w_i}\right),$$
$$PFWG\,(p \oslash p_1, p \oslash p_2, \ldots, p \oslash p_n) = \left(\prod_{i=1}^n \left(\frac{\mu}{\mu_i}\right)^{w_i}, \prod_{i=1}^n \left(\sqrt{\frac{\nu^2-\nu_i^2}{1-\nu_i^2}}\right)^{w_i}\right).$$

Now, $\nu_i\pi \leq \nu\pi_i \Rightarrow \nu_i^2\pi^2 \leq \nu^2\pi_i^2 \Rightarrow \nu_i^2(\nu^2+\pi^2) \leq \nu^2(\nu_i^2+\pi_i^2) \Rightarrow \nu_i^2(1-\mu^2) \leq \nu^2(1-\mu_i^2) \Rightarrow (\mu_i^2-\mu^2)\nu^2 + \nu_i^2(1-\mu^2) \leq (1-\mu^2)\nu^2 \Rightarrow \frac{\mu_i^2-\mu^2}{1-\mu^2} + \frac{\nu_i^2}{\nu^2} \leq 1$.
Similarly, $\frac{\nu^2-\nu_i^2}{1-\nu_i^2} + \frac{\mu^2}{\mu_i^2} \leq 1$.
Based on Lemma 1, we can prove the constraint condition,

$$\left(\prod_{i=1}^n \left(\sqrt{\frac{\mu_i^2-\mu^2}{1-\mu^2}}\right)^{w_i}\right)^2 + \left(\prod_{i=1}^n \left(\frac{\nu_i}{\nu}\right)^{w_i}\right)^2 = \prod_{i=1}^n \left(\frac{\mu_i^2-\mu^2}{1-\mu^2}\right)^{w_i} + \prod_{i=1}^n \left(\frac{\nu_i^2}{\nu^2}\right)^{w_i}$$

$$\leq \sum_{i=1}^{n} w_i \frac{\mu_i^2 - \mu^2}{1-\mu^2} + \sum_{i=1}^{n} w_i \frac{\nu_i^2}{\nu^2} = \sum_{i=1}^{n} w_i \left(\frac{\mu_i^2 - \mu^2}{1-\mu^2} + \frac{\nu_i^2}{\nu^2} \right) \leq \sum_{i=1}^{n} w_i = 1$$

Similarly, $(\prod_{i=1}^{n} (\frac{\mu}{\mu_i})^{w_i})^2 + (\prod_{i=1}^{n} \left(\sqrt{\frac{\nu^2-\nu_i^2}{1-\nu_i^2}}\right)^{w_i})^2 \leq 1.$

From the proof in (1), we can easily get

$$\sqrt{\frac{\mu_i^2 - \mu^2}{1-\mu^2}} \geq \frac{\mu}{\mu_i} \Rightarrow \left(\sqrt{\frac{\mu_i^2 - \mu^2}{1-\mu^2}}\right)^{w_i} \geq \left(\frac{\mu}{\mu_i}\right)^{w_i}$$

$$\Rightarrow \prod_{i=1}^{n} \left(\sqrt{\frac{\mu_i^2 - \mu^2}{1-\mu^2}}\right)^{w_i} \geq \prod_{i=1}^{n} \left(\frac{\mu}{\mu_i}\right)^{w_i},$$

$$\sqrt{\frac{\nu^2 - \nu_i^2}{1-\nu_i^2}} \geq \frac{\nu_i}{\nu} \Rightarrow \left(\sqrt{\frac{\nu^2 - \nu_i^2}{1-\nu_i^2}}\right)^{w_i} \geq \left(\frac{\nu_i}{\nu}\right)^{w_i} \Rightarrow \prod_{i=1}^{n} \left(\sqrt{\frac{\nu^2 - \nu_i^2}{1-\nu_i^2}}\right)^{w_i} \geq \prod_{i=1}^{n} \left(\frac{\nu_i}{\nu}\right)^{w_i}.$$

Thus, according to Definition 3, the proof is proved. □

THEOREM 11. *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs, $p = (\mu, \nu)$ is also PFN, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} w_i = 1$, then*

(1) PFWA $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ PFWA $(p_1, p_2, \ldots, p_n) \otimes p$;
(2) PFWA $(p_1, p_2, \ldots, p_n) \oplus p \geq$ PFWA $(p_1, p_2, \ldots, p_n) \otimes p$;
(3) PFWG $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ PFWG $(p_1, p_2, \ldots, p_n) \otimes p$;
(4) PFWG $(p_1, p_2, \ldots, p_n) \oplus p \geq$ PFWG $(p_1, p_2, \ldots, p_n) \otimes p$;
(5) PFWPA $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ PFWPA $(p_1, p_2, \ldots, p_n) \otimes p$;
(6) PFWPA $(p_1, p_2, \ldots, p_n) \oplus p \geq$ PFWPA $(p_1, p_2, \ldots, p_n) \otimes p$;
(7) PFWPG $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) \geq$ PFWPG $(p_1, p_2, \ldots, p_n) \otimes p$;
(8) PFWPG $(p_1, p_2, \ldots, p_n) \oplus p \geq$ PFWPG $(p_1, p_2, \ldots, p_n) \otimes p$;

Proof. In the following, we shall prove the (1) and the (2)–(8) can be proved analogously.

(1) For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we can get

$$\sqrt{\mu_i^2 + \mu^2 - \mu_i^2\mu^2} \geq \sqrt{2\mu_i^2\mu^2 - \mu_i^2\mu^2} = \mu_i\mu,$$

$$\sum_{i=1}^{n} w_i \sqrt{\mu_i^2 + \mu^2 - \mu_i^2\mu^2} \geq \sum_{i=1}^{n} w_i \mu_i \mu.$$

Similarly, $\sqrt{(\sum_{i=1}^{n} w_i\nu_i)^2 + \nu^2 - (\sum_{i=1}^{n} w_i\nu_i)^2\nu^2} \geq \sum_{i=1}^{n} w_i\nu_i\nu.$

Since *PFWA* $(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p) = (\sum_{i=1}^{n} w_i \sqrt{\mu_i^2 + \nu^2 - \mu_i^2\mu^2},$

$\sum_{i=1}^{n} w_i \nu_i \nu)$,

$$PFWA\,(p_1, p_2, \ldots, p_n) \otimes p = \left(\sum_{i=1}^{n} w_i \mu_i \mu, \sqrt{\left(\sum_{i=1}^{n} w_i \nu_i\right)^2 + \nu^2 - \left(\sum_{i=1}^{n} w_i \nu_i\right)^2 \nu^2} \right).$$

According to Definition 3, the proof is proved. □

THEOREM 12. *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$ be a collection of PFNs, $p = (\mu, \nu)$ is also PFN, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} w_i = 1$, $\lambda \geq 1$, then*

(1) $PFWA\,(\lambda p_1, \lambda p_2, \ldots, \lambda p_n) \geq PFWA\,(p_1^\lambda, p_2^\lambda, \ldots, p_n^\lambda)$;
(2) $\lambda PFWA\,(p_1, p_2, \ldots, p_n) \geq (PFWA\,(p_1, p_2, \ldots, p_n))^\lambda$;
(3) $PFWG\,(\lambda p_1, \lambda p_2, \ldots, \lambda p_n) \geq PFWA\,(p_1^\lambda, p_2^\lambda, \ldots, p_n^\lambda)$;
(4) $\lambda PFWG\,(p_1, p_2, \ldots, p_n) \geq (PFWG\,(p_1, p_2, \ldots, p_n))^\lambda$;
(5) $PFWPA\,(\lambda p_1, \lambda p_2, \ldots, \lambda p_n) \geq PFWA\,(p_1^\lambda, p_2^\lambda, \ldots, p_n^\lambda)$;
(6) $\lambda PFWPA\,(p_1, p_2, \ldots, p_n) \geq (PFWA\,(p_1, p_2, \ldots, p_n))^\lambda$;
(7) $PFWPG\,(\lambda p_1, \lambda p_2, \ldots, \lambda p_n) \geq PFWPG\,(p_1^\lambda, p_2^\lambda, \ldots, p_n^\lambda)$;
(8) $\lambda PFWPG\,(p_1, p_2, \ldots, p_n) \geq (PFWPG\,(p_1, p_2, \ldots, p_n))^\lambda$;

Proof. In the following, we shall prove the (1) and the (2)–(8) can be proved analogously.

(1) For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we have,

$$PFWA\,(\lambda p_1, \lambda p_2, \ldots, \lambda p_n) = \left(\sum_{i=1}^{n} w_i \sqrt{1 - \left(1 - \mu_i^2\right)^\lambda}, \sum_{i=1}^{n} w_i \nu_i^\lambda \right),$$

$$PFWA\,(p_1^\lambda, p_2^\lambda, \ldots, p_n^\lambda) = \left(\sum_{i=1}^{n} w_i \mu_i^\lambda, \sum_{i=1}^{n} w_i \sqrt{1 - \left(1 - \nu_i^2\right)^\lambda} \right).$$

In the following, we denote $f(\mu_i) = 1 - (1 - \mu_i^2)^\lambda - (\mu_i^2)^\lambda$ and prove that $f(\mu_i) \geq 0$. Based on Newton generalized binomial theorem, we can get $(1 - \mu_i^2)^\lambda + (\mu_i^2)^\lambda \leq (1 - \mu_i^2 + \mu_i^2)^\lambda = 1$. Thus, $f(\mu_i) \geq 0$, i.e., $1 - (1 - \mu_i^2)^\lambda \geq (\mu_i^2)^\lambda \Rightarrow \sqrt{1 - (1 - \mu_i^2)^\lambda} \geq \mu_i^\lambda \Rightarrow \sum_{i=1}^{n} w_i \sqrt{1 - (1 - \mu_i^2)^\lambda} \geq \sum_{i=1}^{n} w_i \mu_i^\lambda$.
Similarly, $\sum_{i=1}^{n} w_i \sqrt{1 - (1 - \nu_i^2)^\lambda} \geq \sum_{i=1}^{n} w_i \nu_i^\lambda$.
According to Definition 3, the proof is proved. □

THEOREM 13. *Let $p_i = (\mu_{pi}, \nu_{pi})$ and $q_i = (\mu_{qi}, \nu_{qi})(i = 1, 2, \ldots, n)$ be two collections of PFNs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} w_i = 1$, then*

(1) $PFWA\,(p_1 \oplus q_1, p_2 \oplus q_2, \ldots, p_n \oplus q_n) \geq PFWA\,(p_1 \otimes q_1, p_2 \otimes q_2, \ldots, p_n \otimes q_n)$;
(2) $PFWA\,(p_1, p_2, \ldots, p_n) \oplus PFWA\,(q_1, q_2, \ldots, q_n) \geq PFWA\,(p_1, p_2, \ldots, p_n) \otimes PFWA\,(q_1, q_2, \ldots, q_n)$;
(3) $PFWG\,(p_1 \oplus q_1, p_2 \oplus q_2, \ldots, p_n \oplus q_n) \geq PFWG\,(p_1 \otimes q_1, p_2 \otimes q_2, \ldots, p_n \otimes q_n)$;
(4) $PFWG\,(p_1, p_2, \ldots, p_n) \oplus PFWG\,(q_1, q_2, \ldots, q_n) \geq PFWG\,(p_1, p_2, \ldots, p_n) \otimes PFWG\,(q_1, q_2, \ldots, q_n)$;
(5) $PFWPA\,(p_1 \oplus q_1, p_2 \oplus q_2, \ldots, p_n \oplus q_n) \geq PFWPA\,(p_1 \otimes q_1, p_2 \otimes q_2, \ldots, p_n \otimes q_n)$;

(6) *PFWPA* $(p_1, p_2, \ldots, p_n) \oplus$ *PFWPA* $(q_1, q_2, \ldots, q_n) \geq$ *PFWPA* $(p_1, p_2, \ldots, p_n) \otimes$ *PFWPA* $(q_1, q_2, \ldots, q_n)$;
(7) *PFWPG* $(p_1 \oplus q_1, p_2 \oplus q_2, \ldots, p_n \oplus q_n) \geq$ *PFWPG* $(p_1 \otimes q_1, p_2 \otimes q_2, \ldots, p_n \otimes q_n)$;
(8) *PFWPG* $(p_1, p_2, \ldots, p_n) \oplus$ *PFWPG* $(q_1, q_2, \ldots, q_n) \geq$ *PFWPG* $(p_1, p_2, \ldots, p_n) \otimes$ *PFWPG* $(q_1, q_2, \ldots, q_n)$.

Proof. In the following, we shall prove the (1) and the (2)–(8) can be proved analogously.

(1) For any $p_i = (\mu_{pi}, \nu_{pi}), q_i = (\mu_{qi}, \nu_{qi})(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned} &PFWA\,(p_1 \oplus q_1, p_2 \oplus q_2, \ldots, p_n \oplus q_n) \\ &= \left(\sum_{i=1}^{n} w_i \sqrt{\mu_{pi}^2 + \mu_{qi}^2 - \mu_{pi}^2\mu_{qi}^2}, \sum_{i=1}^{n} w_i \nu_{pi}\nu_{qi}\right), \\ &PFWA\,(p_1 \otimes q_1, p_2 \otimes q_2, \ldots, p_n \otimes q_n) \\ &= \left(\sum_{i=1}^{n} w_i \mu_{pi}\mu_{qi}, \sum_{i=1}^{n} w_i \sqrt{\nu_{pi}^2 + \nu_{qi}^2 - \nu_{pi}^2\nu_{qi}^2}\right). \end{aligned}$$

Because $\mu_{pi}^2 + \mu_{qi}^2 - \mu_{pi}^2\mu_{qi}^2 - \mu_{pi}^2\mu_{qi}^2 = \mu_{pi}^2 + \mu_{qi}^2 - 2\mu_{pi}^2\mu_{qi}^2 \geq 2\mu_{pi}\mu_{qi} - 2\mu_{pi}^2 \mu_{qi}^2 = 2\mu_{pi}\mu_{qi}(1 - \mu_{pi}\mu_{qi}) \geq 0$, so we can obtain $\mu_{pi}^2 + \mu_{qi}^2 - \mu_{pi}^2\mu_{qi}^2 \geq \mu_{pi}^2\mu_{qi}^2$, i.e., $\sum_{i=1}^{n} w_i \sqrt{\mu_{pi}^2 + \mu_{qi}^2 - \mu_{pi}^2\mu_{qi}^2} \geq \sum_{i=1}^{n} w_i \mu_{pi}\mu_{qi}$.
Similarly, $\sum_{i=1}^{n} w_i \sqrt{\nu_{pi}^2 + \nu_{qi}^2 - \nu_{pi}^2\nu_{qi}^2} \geq \sum_{i=1}^{n} w_i \nu_{pi}\nu_{qi}$.
According to Definition 3, the proof is proved. □

THEOREM 14. *Let* $p_i = (\mu_{pi}, \nu_{pi})$ *and* $q_i = (\mu_{qi}, \nu_{qi})(i = 1, 2, \ldots, n)$ *be two collections of PFNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of them,* $\sum_{i=1}^{n} w_i = 1$*, and* $\lambda \geq 1$*, then*

(1) *PFWA* $(\lambda p_1 \oplus p, \lambda p_2 \oplus p, \ldots, \lambda p_n \oplus p) \geq$ *PFWA* $(p_1^\lambda \otimes p, p_2^\lambda \otimes p, \ldots, p_n^\lambda \otimes p)$;
(2) λ*PFWA* $(p_1, p_2, \ldots, p_n) \oplus p \geq ($*PFWA* $(p_1, p_2, \ldots, p_n))^\lambda \otimes p$;
(3) *PFWG* $(\lambda p_1 \oplus p, \lambda p_2 \oplus p, \ldots, \lambda p_n \oplus p) \geq$ *PFWG* $(p_1^\lambda \otimes p, p_2^\lambda \otimes p, \ldots, p_n^\lambda \otimes p)$;
(4) λ*PFWG* $(p_1, p_2, \ldots, p_n) \oplus p \geq ($*PFWG* $(p_1, p_2, \ldots, p_n))^\lambda \otimes p$;
(5) *PFWPA* $(\lambda p_1 \oplus p, \lambda p_2 \oplus p, \ldots, \lambda p_n \oplus p) \geq$ *PFWPA* $(p_1^\lambda \otimes p, p_2^\lambda \otimes p, \ldots, p_n^\lambda \otimes p)$;
(6) λ*PFWPA* $(p_1, p_2, \ldots, p_n) \oplus p \geq ($*PFWPA* $(p_1, p_2, \ldots, p_n))^\lambda \otimes p$;
(7) *PFWPG* $(\lambda p_1 \oplus p, \lambda p_2 \oplus p, \ldots, \lambda p_n \oplus p) \geq$ *PFWPG* $(p_1^\lambda \otimes p, p_2^\lambda \otimes p, \ldots, p_n^\lambda \otimes p)$;
(8) λ*PFWPG* $(p_1, p_2, \ldots, p_n) \oplus p \geq ($*PFWPG* $(p_1, p_2, \ldots, p_n))^\lambda \otimes p$;

Proof. In the following, we shall prove the (1), and the (2)–(8) can be proved analogously.

(1) For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we have,

$$\begin{aligned} &PFWA\,(\lambda p_1 \oplus p, \lambda p_2 \oplus p, \ldots, \lambda p_n \oplus p) \\ &= \left(\sum_{i=1}^{n} w_i \sqrt{1 - (1 - \mu^2)\left(1 - \mu_i^2\right)^\lambda}, \sum_{i=1}^{n} w_i \nu_i^\lambda \nu\right), \end{aligned}$$

$$PFWA\left(p_1^{\lambda}\otimes p, p_2^{\lambda}\otimes p, \ldots, p_n^{\lambda}\otimes p\right)$$
$$=\left(\sum_{i=1}^{n} w_i \mu_i^{\lambda}\mu, \sum_{i=1}^{n} w_i \sqrt{1-(1-\nu^2)\left(1-\nu_i^2\right)^{\lambda}}\right)$$

In the following, we denote $f(\mu_i)=1-(1-\mu_i^2)^{\lambda}(1-\mu^2)-(\mu_i^2)^{\lambda}\mu^2$ and prove that $f(\mu_i)\geq 0$. First, we denote $g(\mu_i)=(\mu_i^2)^{\lambda}+(1-\mu_i^2)^{\lambda}$, and take the derivative of $g(\mu_i)$ and obtain $g'(\mu_i)=2\lambda\mu_i((\mu_i^2)^{\lambda-1}-(1-\mu_i^2)^{\lambda-1})$, therefore, if $\mu_i>\frac{\sqrt{2}}{2}$, $g(\mu_i)$ is monotonically increasing, and if $\mu_i<\frac{\sqrt{2}}{2}$, $g(\mu_i)$ is monotonically decreasing, so $g(\mu_i)\leq g(\mu_i)_{\max}=\max\{g(0), g(1)\}=1$. Because $(1-\mu_i^2)^{\lambda}(1-\mu^2)+(\mu_i^2)^{\lambda}\mu^2\leq 1$, hence $f(\mu_i)=1-(1-\mu_i^2)^{\lambda}(1-\mu^2)-(\mu_i^2)^{\lambda}\mu^2\geq 0 \Rightarrow \sum_{i=1}^{n} w_i\sqrt{1-(1-\mu^2)(1-\mu_i^2)^{\lambda}}\geq \sum_{i=1}^{n} w_i\mu_i^{\lambda}\mu$.
Similarly, $\sum_{i=1}^{n} w_i\sqrt{1-(1-\nu^2)(1-\nu_i^2)^{\lambda}}\geq \sum_{i=1}^{n} w_i\nu_i^{\lambda}\nu$.
According to Definition 3, the proof is proved. □

THEOREM 15. *Let $p_i=(\mu_i,\nu_i)$ be a collection of PFNs and $w=(w_1, w_2, \ldots, w_n)^T$ be the weight vector of them, $\sum_{i=1}^{n} w_i=1$, then*

(1) $PFWA\,(p_1^c, p_2^c, \ldots, p_n^c)=(PFWA\,(p_1, p_2, \ldots, p_n))^c$;
(2) $PFWG\,(p_1^c, p_2^c, \ldots, p_n^c)=(PFWG\,(p_1, p_2, \ldots, p_n))^c$;
(3) $PFWPA\,(p_1^c, p_2^c, \ldots, p_n^c)=(PFWPA\,(p_1, p_2, \ldots, p_n))^c$;
(4) $PFWPG\,(p_1^c, p_2^c, \ldots, p_n^c)=(PFWPG\,(p_1, p_2, \ldots, p_n))^c$.

Proof. In the following, we shall prove the (1) and the (2)–(4) can be proved analogously.

(1) For any $p_i=(\mu_i,\nu_i)(i=1,2,\ldots,n)$, we have,

$$PFWA\left(p_1^c, p_2^c, \ldots, p_n^c\right)=\left(\sum_{i=1}^{n} w_i\nu_i, \sum_{i=1}^{n} w_i\mu_i\right)$$
$$(PFWA\,(p_1, p_2, \ldots, p_n))^c=\left(\sum_{i=1}^{n} w_i\mu_i, \sum_{i=1}^{n} w_i\nu_i\right)^c$$
$$=\left(\sum_{i=1}^{n} w_i\nu_i, \sum_{i=1}^{n} w_i\mu_i\right)=PFWA\left(p_1^c, p_2^c, \cdots, p_n^c\right)$$

□

THEOREM 16 (Boundedness). *Let $p_i=(\mu_i,\nu_i)(i=1,2,\ldots,n)$ be a collection of PFNs and $w=(w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p_i(i=1,2,\ldots,n)$, with $\sum_{i=1}^{n}=1$. Assume that $\mu^-=\min\limits_{1\leq i\leq n}\{\mu_i\}$, $\mu^+=\max\limits_{1\leq i\leq n}\{\mu_i\}$, $\nu^-=\min\limits_{1\leq i\leq n}\{\nu_i\}$, $\nu^+=\max\limits_{1\leq i\leq n}\{\nu_i\}$, then we can get*

(1) $(\mu^-,\nu^+)\leq PFWA\,(p_1, p_2, \ldots, p_n)\leq(\mu^+,\nu^-)$;
(2) $(\mu^-,\nu^+)\leq PFWG\,(p_1, p_2, \ldots, p_n)\leq(\mu^+,\nu^-)$;
(3) $(\mu^-,\nu^+)\leq PFWPA\,(p_1, p_2, \ldots, p_n)\leq(\mu^+,\nu^-)$;
(4) $(\mu^-,\nu^+)\leq PFWPG\,(p_1, p_2, \ldots, p_n)\leq(\mu^+,\nu^-)$.

Proof. In the following, we shall prove the (1), (3) and the (2), (4) can be proved analogously.

For any $p_i = (\mu_i, \nu_i)(i = 1, 2, \ldots, n)$, we can get $\mu^- \leq \mu_i \leq \mu^+$, $\nu^- \leq \nu_i \leq \nu^+(i = 1, 2, \ldots, n)$. Suppose that $p_{\min} = (\mu^-, \nu^+)$, $p_{\max} = (\mu^+, \nu^-)$.

(1) $\sum_{i=1}^n w_i\mu^- \leq \sum_{i=1}^n w_i\mu_i \leq \sum_{i=1}^n w_i\mu^+$, $\sum_{i=1}^n w_i\nu^- \leq \sum_{i=1}^n w_i\nu_i \leq \sum_{i=1}^n w_i\nu^+$.

$$s(p_{\min}) = (\mu^-)^2 - (\nu^+)^2 = \left(\sum_{i=1}^n w_i\mu^-\right)^2 - \left(\sum_{i=1}^n w_i\nu^+\right)^2,$$

$$s(p_{\max}) = (\mu^+)^2 - (\nu^-)^2 = \left(\sum_{i=1}^n w_i\mu^+\right)^2 - \left(\sum_{i=1}^n w_i\nu^-\right)^2,$$

$$s(PFWA\,(p_1, p_2, \ldots, p_n)) = \left(\sum_{i=1}^n w_i\mu_i\right)^2 - \left(\sum_{i=1}^n w_i\nu_i\right)^2.$$

Consequently, $s(p_{\min}) \leq s(PFWA) \leq s(p_{\max})$.
Therefore, $(\mu^-, \nu^+) \leq PFWA\,(p_1, p_2, \ldots, p_n) \leq (\mu^+, \nu^-)$.

(3) $\mu^- = (\sum_{i=1}^n w_i(\mu^-)^2)^{1/2}$, $\mu^+ = (\sum_{i=1}^n w_i(\mu^+)^2)^{1/2}$, $\nu^+ = (\sum_{i=1}^n w_i(\nu^+)^2)^{1/2}$, $\nu^- = (\sum_{i=1}^n w_i(\nu^-)^2)^{1/2}$.
Hence,

$$\left(\sum_{i=1}^n w_i(\mu^-)^2\right)^{1/2} \leq \left(\sum_{i=1}^n w_i(\mu_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^n w_i(\mu^+)^2\right)^{1/2},$$

$$\left(\sum_{i=1}^n w_i(\nu^-)^2\right)^{1/2} \leq \left(\sum_{i=1}^n w_i(\nu_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^n w_i(\nu^+)^2\right)^{1/2},$$

$$s(p_{\min}) = (\mu^-)^2 - (\nu^+)^2 = \left(\sum_{i=1}^n w_i(\mu^-)^2\right)^{1/2} - \left(\sum_{i=1}^n w_i(\nu^+)^2\right)^{1/2},$$

$$s(p_{max}) = (\mu^+)^2 - (\nu^-)^2 = \left(\sum_{i=1}^n w_i(\mu^+)^2\right)^{1/2} - \left(\sum_{i=1}^n w_i(\nu^-)^2\right)^{1/2},$$

$$s(PFWA\,(p_1, p_2, \ldots, p_n)) = \left(\sum_{i=1}^n w_i(\mu_i)^2\right)^{1/2} - \left(\sum_{i=1}^n w_i(\nu_i)^2\right)^{1/2}.$$

Consequently, $s(p_{\min}) \leq s(PFWPA) \leq s(p_{max})$.
Therefore, $(\mu^-, \nu^+) \leq PFWPA\,(p_1, p_2, \ldots, p_n) \leq (\mu^+, \nu^-)$. □

THEOREM 17 (Idempotency). *If all $p_i(i = 1, 2, \ldots, n)$ are equal, and $p_i = p = (\mu, \nu)$, with $\sum_{i=1}^n w_i = 1$, then*

(1) $PFWA\,(p_1, p_2, \ldots, p_n) = p$;
(2) $PFWG\,(p_1, p_2, \ldots, p_n) = p$;
(3) $PFWPA\,(p_1, p_2, \ldots, p_n) = p$;
(4) $PFWPG\,(p_1, p_2, \ldots, p_n) = p$.

Proof. In the following, we shall prove the (1), (3) and the (2), (4) can be proved analogously.

Since $p_i = p = (\mu, \nu)$ for all $i = 1, 2, \ldots, n$, then

(1) $PFWA\ (p_1, p_2, \ldots, p_n) = PFWA\ (p, p, \ldots, p) = (\sum_{i=1}^{n} w_i\mu, \sum_{i=1}^{n} w_i\nu) = (\mu, \nu) = p$.

(3) $PFWPA\ (p_1, p_2, \ldots, p_n) = PFWPA\ (p, p, \ldots, p) = ((\sum_{i=1}^{n} w_i\mu^2)^{1/2}, (\sum_{i=1}^{n} w_i\nu^2)^{1/2}) = (\mu, \nu) = p$.

□

THEOREM 18 (Monotonicity). *Let $p_i = (\mu_{p_i}, \nu_{p_i})$ and $q_i = (\mu_{q_i}, \nu_{q_i})(i = 1, 2, \ldots, n)$ be two collections of PFNs, if $\mu_{q_i} \geq \mu_{p_i}, \nu_{q_i} \leq \nu_{p_i}$ for all i, then*

(1) $PFWA\ (p_1, p_2, \ldots, p_n) \leq PFWA\ (q_1, q_2, \ldots, q_n)$;
(2) $PFWG\ (p_1, p_2, \ldots, p_n) \leq PFWG\ (q_1, q_2, \ldots, q_n)$;
(3) $PFWPA\ (p_1, p_2, \ldots, p_n) \leq PFWPA\ (q_1, q_2, \ldots, q_n)$;
(4) $PFWPG\ (p_1, p_2, \cdots, p_n) \leq PFWPG\ (q_1, q_2, \cdots, q_n)$.

Proof. In the following, we shall prove the (1), (3) and the (2), (4) can be proved analogously.

Since $\mu_{q_i} \geq \mu_{p_i}, \nu_{q_i} \leq \nu_{p_i}$ for all i, then

(1) $\sum_{i=1}^{n} w_i\mu_{p_i} \leq \sum_{i=1}^{n} w_i\mu_{q_i}, \sum_{i=1}^{n} w_i\nu_{q_i} \leq \sum_{i=1}^{n} w_i\nu_{p_i}$.
Hence,

$$s(PFWA\ (p_1, p_2, \ldots, p_n)) = \left(\sum_{i=1}^{n} w_i\mu_{p_i}\right)^2 - \left(\sum_{i=1}^{n} w_i\nu_{p_i}\right)^2,$$

$$s(PFWA\ (q_1, q_2, \ldots, q_n)) = \left(\sum_{i=1}^{n} w_i\mu_{q_i}\right)^2 - \left(\sum_{i=1}^{n} w_i\nu_{q_i}\right)^2.$$

Apparently, $s(PFWA\ (p_1, p_2, \ldots, p_n)) \leq s(PFWA\ (q_1, q_2, \ldots, q_n))$.
Consequently, $PFWA\ (p_1, p_2, \ldots, p_n) \leq PFWA\ (q_1, q_2, \ldots, q_n)$.

(3) $\sum_{i=1}^{n} w_i(\mu_{p_i})^2 \leq \sum_{i=1}^{n} w_i(\mu_{q_i})^2, \sum_{i=1}^{n} w_i(\nu_{q_i})^2 \leq \sum_{i=1}^{n} w_i(\nu_{p_i})^2$.
Hence,

$$\begin{aligned} s(PFWPA\ (p_1, p_2, \ldots, p_n)) &= \left(\left(\sum_{i=1}^{n} w_i(\mu_{p_i})^2\right)^{1/2}\right)^2 - \left(\left(\sum_{i=1}^{n} w_i(\nu_{p_i})^2\right)^{1/2}\right)^2 \\ &= \sum_{i=1}^{n} w_i(\mu_{p_i})^2 - \sum_{i=1}^{n} w_i(\nu_{p_i})^2 \\ s(PFWPA\ (q_1, q_2, \ldots, q_n)) &= \left(\left(\sum_{i=1}^{n} w_i(\mu_{q_i})^2\right)^{1/2}\right)^2 - \left(\left(\sum_{i=1}^{n} w_i(\nu_{q_i})^2\right)^{1/2}\right)^2 \\ &= \sum_{i=1}^{n} w_i(\mu_{q_i})^2 - \sum_{i=1}^{n} w_i(\nu_{q_i})^2 \end{aligned}$$

Apparently, $s(PFWPA\ (p_1, p_2, \ldots, p_n)) \leq s(PFPWA\ (q_1, q_2, \ldots, q_n))$.
Consequently, $PFWPA\ (p_1, p_2, \ldots, p_n) \leq PFPWA\ (q_1, q_2, \ldots, q_n)$. □

5. A PF-SIR APPROACH TO MAGDM WITH PYTHAGOREAN FUZZY INFORMATION

Assume that in a Pythagorean fuzzy group decision-making problem, the weights and the attribute values all take the form of Pythagorean fuzzy information. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a discrete set of alternatives, and $C = \{c_1, c_2, \ldots, c_n\}$ be a finite set of attributes. Suppose that $E = \{e_1, e_2, \ldots, e_l\}$ be a set of experts with weight vector $W = \{W_1, W_2, \ldots, W_l\}$. Assume that $P(k) = (p_{ij}^{(k)})_{m\times n}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, l)$ is the Pythagorean fuzzy decision matrix, where $p_{ij}^{(k)}$ denotes the attribute value that alternative x_i satisfies the attribute c_j given by expert e_k. The attribute weights decision matrix is $w = (w_j^{(k)})_{l\times n}$, where $w_j^{(k)}$ denotes that the weight value of the attribute c_j given by expert e_k.

In the following, we will propose a novel approach based on PF-SIR with Pythagorean fuzzy information.

Step 1. Determine the individual measure degree $\xi_k (k = 1, 2, \ldots, l)$ via the weights of experts, which take the form of PFEs. The relative closeness coefficient is obtained as follows:

$$\xi_k = \frac{d(W_k, W^-)}{d(W_k, W^-) + d(W_k, W^+)}, \tag{11}$$

where $W^- = (\min_k\{\mu_k\}, \max_k\{\nu_k\})$, $W^+ = (\max_k\{\mu_k\}, \min_k\{\nu_k\})$. It is easily obtained that $0 \leq W_k \leq 1$ and if $\xi_k \to 1$, then $W_k \to W^+$; if $\xi_k \to 0$, then $W_k \to W^-$.

Step 2. Normalize the $\xi_k (k = 1, 2, \ldots, l)$ to make their sum into a unit and get as follows:

$$\zeta_k = \frac{\xi_k}{\sum_{k=1}^{l} \xi_k}, \tag{12}$$

we get the normalized vector of real numbers $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_l)^T$ as individual measure degrees.

Step 3. Utilize the PFWA operator to aggregate individual viewpoints into group viewpoints as follows:

(a) individual decision matrix integration:

$$\overline{p}_{ij} = PFWA_{\zeta_k}\left(p_{ij}^{(1)}, p_{ij}^{(2)}, \ldots, p_{ij}^{(l)}\right) = \left(\sum_{k=1}^{l} \zeta_k \mu_{ij}^{(k)}, \sum_{k=1}^{l} \zeta_k \nu_{ij}^{(k)}\right) \tag{13}$$

(b) individual attributes' weights integration:

$$\overline{w}_j = \mathrm{PFWA}_{\zeta_k}\left(w_j^{(1)}, w_j^{(2)}, \ldots, w_j^{(l)}\right) = \left(\sum_{k=1}^{l} \zeta_k u_j^{(k)}, \sum_{k=1}^{l} \zeta_k v_j^{(k)}\right) \tag{14}$$

From this step, the group integrated decision matrix $p = (\overline{p}_{ij})_{m\times n}$ and the attribute weights vector $\overline{w} = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$ are achieved.

Step 4. Obtain Pythagorean fuzzy superiority/inferiority matrix

(a) Obtain the performance function f_{ij}

$$f_{ij} = f(\overline{p}_{ij}) = \frac{d(\overline{p}_{ij}, \overline{p}^-)}{d(\overline{p}_{ij}, \overline{p}^-) + d(\overline{p}_{ij}, \overline{p}^+)}, \tag{15}$$

where $\overline{p}^- = \{c_j, (\min_i\{\mu_{ij}\}, \max_i\{\nu_{ij}\})\}$, $\overline{p}^+ = \{c_j, (\max_i\{\mu_{ij}\}, \min_i\{\nu_{ij}\})\}$. It is easily obtained that $0 \leq f_{ij} \leq 1$ and if $f_{ij} \to 1$, then $\overline{p}_{ij} \to \overline{p}^+$; if $f_{ij} \to 0$, then $\overline{p}_{ij} \to \overline{p}^-$.

(b) Obtain the preference intensity $PI_j(x_i, x_t)$

We define $PI_j(x_i, x_t)(t, i = 1, 2, \ldots, m, t \neq i; j = 1, 2, \ldots, n)$ as the preference intensity of alternative x_i over alternative x_t to the corresponding attribute c_j, which is given as follows:

$$PI_j(x_i, x_t) = \phi_j(f_{ij} - f_{tj}) = \phi_j(d), \tag{16}$$

where $\phi_j(d)$ is a nondecreasing function from the real number to [0,1]. Normally, $\phi_j(d)$ can be chosen from six generalized threshold functions,[15] or defined by the experts themselves.

(c) Obtain superiority matrix and inferiority matrix

Superiority index (S-index): $S = (S_{ij})_{m\times n}$

$$S_{ij} = \sum_{t=1}^{n} PI_j(x_i, x_t) = \sum_{t=1}^{n} \phi_j(f_{ij} - f_{tj}); \tag{17}$$

Inferiority index (I-index): $I = (I_{ij})_{m\times n}$

$$I_{ij} = \sum_{t=1}^{n} PI_j(x_t, x_i) = \sum_{t=1}^{n} \phi_j(f_{tj} - f_{ij}); \tag{18}$$

Step 5. Compute the superiority flow and inferiority flow as follows:

S-flow

$$\phi^{>}(x_i) = PFWA_{S_{ij}}(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n) = \left(\sum_{j=1}^{n}\left(S_{ij}\sum_{k=1}^{l}\zeta_k u_j^{(k)}\right), \sum_{j=1}^{n}\left(S_{ij}\sum_{k=1}^{l}\zeta_k v_j^{(k)}\right)\right) \tag{19}$$

I-flow

$$\phi^{<}(x_i) = PFWA_{I_{ij}}(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n) = \left(\sum_{j=1}^{n}\left(I_{ij}\sum_{k=1}^{l}\zeta_k u_j^{(k)}\right), \sum_{j=1}^{n}\left(I_{ij}\sum_{k=1}^{l}\zeta_k v_j^{(k)}\right)\right) \tag{20}$$

Then, according to Equation 5, we compute the score function of corresponding S-flow $\phi^{>}(x_i)$ and I-flow $\phi^{<}(x_i)$, respectively. Hence, we get S-flow and I-flow of alternative

x_i as $x_i(\phi^>(x_i), \phi^<(x_i))$. Apparently, if the bigger S-flow $\phi^>(x_i)$ and the smaller I-flow $\phi^<(x_i)$, the alternative x_i is better.

Step 6. Superiority ranking Rule (SR-Rule):

SR-Rule 1. If $\phi^>(x_i) > \phi^>(x_t)$ and $\phi^<(x_i) < \phi^<(x_t)$, then $x_i \succ x_t$;
SR-Rule 2. If $\phi^>(x_i) > \phi^>(x_t)$ and $\phi^<(x_i) = \phi^<(x_t)$, then $x_i \succ x_t$;
SR-Rule 3. If $\phi^>(x_i) = \phi^>(x_t)$ and $\phi^<(x_i) < \phi^<(x_t)$, then $x_i \succ x_t$.

Inferiority ranking Rule (IR-Rule):

IR-Rule 1. If $\phi^>(x_i) < \phi^>(x_t)$ and $\phi^<(x_i) > \phi^<(x_t)$, then $x_i \prec x_t$;
IR-Rule 2. If $\phi^>(x_i) < \phi^>(x_t)$ and $\phi^<(x_i) = \phi^<(x_t)$, then $x_i \prec x_t$;
IR-Rule 3. If $\phi^>(x_i) = \phi^>(x_t)$ and $\phi^<(x_i) > \phi^<(x_t)$, then $x_i \prec x_t$.

Step 7. Combine SR-Rule and IR-Rule, we can get the best alternative $x_i (i = 1, 2, \ldots, m)$.

6. NUMERICAL EXAMPLES

Listed Internet companies play an important role in China's stock market. Performance of listed companies affects resources allocation of capital market and has become a common concern of shareholders, creditors, government authorities, and other stakeholders. An investment bank wants to invest a sum of money in Internet stocks. So the investment bank hires three types of experts: market maker e_1, dealer e_2, and finder e_3 to evaluate the potential investment value. They choose four Internet stocks in which the earnings ratio is higher than other stocks: (1) x_1 is SINA; (2) x_2 is BIDU; (3) x_3 is NETS; (4) x_4 is BABA from three attributes: (1) c_1 is the stock market trend; (2) c_2 is the policy direction; (3) c_3 is the annual performance. The three experts $e_k (k = 1, 2, 3)$ evaluate the Internet stocks $x_i (i = 1, 2, 3, 4)$ with respect to the attributes $c_j (j = 1, 2, 3)$ and construct the following three Pythagorean fuzzy decision matrices $P(k) = (p_{ij}^{(k)})_{4\times 3}$ in Table I. Tables II and III show the

Table I. Pythagorean fuzzy decision matrices.

e_1	c_1	c_2	c_3
x_1	(0.9, 0.3)	(0.8, 0.1)	(0.9, 0.2)
x_2	(0.5, 0.7)	(0.4, 0.7)	(0.8, 0.1)
x_3	(0.3, 0.5)	(0.8, 0.4)	(0.3, 0.8)
x_4	(0.6, 0.7)	(0.5, 0.6)	(0.4, 0.2)
e_2	c_1	c_2	c_3
x_1	(0.7, 0.2)	(0.9, 0.2)	(0.7, 0.2)
x_2	(0.6, 0.7)	(0.5, 0.6)	(0.6, 0.2)
x_3	(0.7, 0.1)	(0.6, 0.5)	(0.8, 0.4)
x_4	(0.6, 0.6)	(0.5, 0.8)	(0.6, 0.4)
e_3	c_1	c_2	c_3
x_1	(0.8, 0.1)	(0.9, 0.2)	(0.8, 0.1)
x_2	(0.7, 0.6)	(0.6, 0.4)	(0.7, 0.2)
x_3	(0.9, 0.4)	(0.8, 0.6)	(0.7, 0.4)
x_4	(0.8, 0.6)	(0.7, 0.5)	(0.6, 0.4)

Table II. The weights of experts.

Experts	*PFEs*
e_1	(0.8, 0.3)
e_2	(0.9, 0.4)
e_3	(0.7, 0.2)

Table III. The weights of attributes.

	c_1	c_2	c_3
e_1	(0.8, 0.3)	(0.7, 0.2)	(0.6, 0.4)
e_2	(0.9, 0.1)	(0.8, 0.3)	(0.8, 0.1)
e_3	(0.6, 0.2)	(0.9, 0.2)	(0.8, 0.2)

Table IV. The PF-SIR flows of Internet stocks.

Internet stocks	$\phi^{>}(x_i)$	$s(\phi^{>}(x_i))$	$\phi^{<}(x_i)$	$s(\phi^{<}(x_i))$
x_1 *(SINA)*	(0.0221, 0.0064)	0.000446	(0, 0)	0
x_2 *(BIDU)*	(0.0147, 0.0043)	0.000198	(0.0074, 0.0021)	0.00005
x_3 *(NETS)*	(0, 0)	0	(0.0221, 0.0064)	0.000446
x_4 *(BABA)*	(0.0074, 0.0021)	0.00005	(0.0147, 0.0043)	0.000198

weights of experts and attribute weights, respectively, which all take the form of PFEs.

Then, we utilize the approach developed in Section 5 to get the most desirable alternative(s), which involves the following steps:

Step 1. Determine the individual measure degree $\xi_k (k = 1, 2, 3)$ using Equation 11:

$$\xi = (0.4688, 0.7273, 0.2727)^T.$$

Step 2. Obtain the normalized vector using Equation 12:

$$\zeta = (0.3191, 0.4952, 0.1857)^T.$$

Step 3. The attribute weights can be obtained as using Equation 13:

$$\overline{w}_1 = (0.8124, 0.1824), \overline{w}_2 = (0.7867, 0.2495), \overline{w}_3 = (0.7362, 0.2143).$$

The aggregated decision values can be obtained as using Equation 14:

$$(\overline{p}_{ij})_{4\times 3} = \begin{pmatrix} (0.7824, 0.2133) & (0.8681, 0.1681) & (0.7824, 0.1814) \\ (0.5867, 0.6814) & (0.4867, 0.5948) & (0.6824, 0.1681) \\ (0.6095, 0.2834) & (0.7010, 0.4867) & (0.6219, 0.5277) \\ (0.6371, 0.6319) & (0.5371, 0.6805) & (0.5362, 0.3362) \end{pmatrix}$$

Step 4. Obtain the performance function f_{ij} using Equation 14:

$$(f_{ij})_{4\times 3} = \begin{pmatrix} 1 & 1 & 0.9859 \\ 0 & 0.1746 & 0.7606 \\ 0.6147 & 0.4926 & 0.2840 \\ 0.1552 & 0.0999 & 0.3376 \end{pmatrix}$$

we set the threshold attribute function as follows:

$$\phi_k(d) = \begin{cases} 0.01 & if \quad d > 0 \\ 0.00 & if \quad d \leq 0 \end{cases}$$

Obtain the superiority matrix (S-matrix) uisng Equation 16:

$$S = \begin{pmatrix} 0.03 & 0.03 & 0.03 \\ 0 & 0.01 & 0.02 \\ 0.02 & 0.02 & 0 \\ 0.01 & 0 & 0.01 \end{pmatrix}$$

Obtain the inferiority matrix (I-matrix) using Equation 17:

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0.03 & 0.02 & 0.01 \\ 0.01 & 0.01 & 0.03 \\ 0.02 & 0.03 & 0.02 \end{pmatrix}$$

Step 5. Compute the superiority flow and inferiority flow using Equations 18 and 19, which are shown in Table IV.

Step 6. Combine Table IV with SR-Rule and the following can be obtained:

$$x_1 \succ x_2 \succ x_4 \succ x_3$$

and, combine Table IV with IR-Rule and the following can be obtained:

$$x_1 \succ x_2 \succ x_4 \succ x_3.$$

Step 7. According to the results of SR-Rule and IR-Rule, the most desirable investment value of Internet stock is x_1(SINA).

7. CONCLUSIONS

In this paper, two new operations subtraction and division over PFNs are proposed. Based on the operations of PFNs, a series of new properties have been discussed in detail. Meanwhile, based on Pythagorean fuzzy aggregation operators, we explore their properties such as boundedness, idempotency, and monotonicity. Finally, we propose a PF-SIR method and apply it to Internet stocks investment. In the future, we will combine others methods with PFSs.

Acknowledgments

The authors are very grateful to the anonymous reviewers for their valuable comments and constructive suggestions, which greatly improved the quality of this paper. Our work is sponsored by the National Science Foundation of China (No. 61163036).

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