Pythagorean Fuzzy Choquet Integral Based MABAC Method for Multiple Attribute Group Decision Making

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In this paper, we define the Choquet integral operator for Pythagorean fuzzy aggregation operators, such as Pythagorean fuzzy Choquet integral average (PFCIA) operator and Pythagorean fuzzy Choquet integral geometric (PFCIG) operator. The operators not only consider the importance of the elements or their ordered positions but also can reflect the correlations among the elements or their ordered positions. It is worth pointing out that most of the existing Pythagorean fuzzy aggregation operators are special cases of our operators. Meanwhile, some basic properties are discussed in detail. Later, we propose two approaches to multiple attribute group decision making with attributes involving dependent and independent by the PFCIA operator and multi-attributive border approximation area comparison (MABAC) in Pythagorean fuzzy environment. Finally, two illustrative examples have also been taken in the present study to verify the developed approaches and to demonstrate their practicality and effectiveness. © 2016 Wiley Periodicals, Inc.

1. INTRODUCTION

Intuitionistic fuzzy set (IFS), initiated by Atanassov,[1] is an extension of fuzzy set theory.[2] IFS is characterized by a membership degree and a nonmembership degree, and therefore can depict the fuzzy character of data more comprehensively and detailedly. To obtain a decision, an important step is the aggregation of intuitionistic fuzzy numbers (IFNs). Intuitionistic fuzzy aggregation operators are the most widely used techniques for aggregating IFNs. In the past decades, a series of intuitionistic fuzzy aggregation operators have been developed.[3,13]

Recently, Yager[14] proposed Pythagorean fuzzy set (PFS) characterized by a membership degree and a nonmembership degree which satisfies the condition that the square sum of its membership degree and nonmembership degree is less than or equal to 1. Yager and Abbasov[15] gave an example to state this situation: A decision maker gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his against membership is $\frac{1}{2}$. Owing to the sum of two values is bigger than 1, they are not

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available for IFS, but they are available for PFS since $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \le 1$. Obviously, PFS is more capable of than IFS to model the vagueness in the practical multiple attribute decision-making (MADM) problems. Based on the modificatory TOPSIS method,[16] Zhang and Xu[17] developed an extension of Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) to MADM with PFS information. Yang et al.[18] pointed out an error to the proof in Zhang and Xu.[17] Yager[19] proposed a series of aggregation operators: Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric average (PFWG) operator, Pythagorean fuzzy weighted power average (PFWPA) operator, and Pythagorean fuzzy weighted power geometric (PFWPG) operator and applied them to MADM problems. Peng and Yang[20] discussed their relationship and proposed a superiority and inferiority ranking (SIR) multiple attribute group decision-making (MAGDM) method. Meanwhile, inspired by soft set theory[21] and linguistic set theory,[22] they proposed Pythagorean fuzzy soft sets[23] and Pythagorean fuzzy linguistic sets,[24] respectively. Gou et al.[25] developed several Pythagorean fuzzy functions and investigated their fundamental properties such as continuity, derivability, and differentiability in detail. Zhang[26] presented a hierarchical qualitative flexible (QUALIFLEX) multiple criteria approach with the closeness index-based ranking methods for multicriteria Pythagorean fuzzy decision analysis. Beliakov and James[27] focused on how the notion of "averaging" should be treated in the case of Pythagorean fuzzy numbers (PFNs) and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat and Yager[28] applied the PFNs in handling the collaborative-based recommender system. Bustince et al.[29] reviewed the definitions and basic properties of the different types of fuzzy sets that have appeared up to now in the literature and pointed out that in any case IFSs are PFSs. Li et al.[30] introduced a knowledge checking service selection method in the Pythagorean fuzzy environment. Peng and Yang[31] extended the Pythagorean fuzzy sets to that of interval-valued Pythagorean fuzzy sets, which can describe the data more accurately and precisely and also developed an interval-valued Pythagorean fuzzy elimination and choice translating reality (ELECTRE) method to solve the MAGDM problem with interval-valued Pythagorean fuzzy numbers.

The aggregation operators proposed by Yager[19] are linear in nature and do not consider the interdependency or interactive characteristics of decision criteria or preferences of decision makers. Sugeno[32] introduced fuzzy measure to model interaction phenomena among the decision criteria[33] and was used in many MADM problems with interdependent decision criteria.[34,35] From the above analysis, we can see that in general, the assumption of independency of criteria is too strong to be satisfied in many MADM and MAGDM problems. Motivated by the Choquet integral,[36,37] some scholars have extended it to solve the decision-making problems with different fuzzy environments, such as in intuitionistic fuzzy environment,[38,40] interval-valued intuitionistic fuzzy environment,[41], hesitant fuzzy environment,[42] multiset hesitant fuzzy environment,[43] dual hesitant fuzzy environment,[44] interval-valued intuitionistic hesitant fuzzy environment.[45] However, all of them fail to the Pythagorean fuzzy environment using the Choquet integral. Therefore, we develop some Pythagorean fuzzy Choquet integral aggregation (PFCIA) operators, such as a Pythagorean fuzzy Choquet integral averaging operator, whose prominent characteristic is that they can not only consider the importance of the elements or their ordered positions but also reflect the correlations of the elements or their ordered positions.

The multiattributive border approximation area comparison (MABAC) method is a new MADM method proposed by Pamucar and Cirovic.[46] It has a simple computation process, systematic procedure, and a sound logic that represents the rationale of human decision making. Hence, it is an interesting research topic to apply the MABAC in a R&D project selection process to rank and determine the best project under the Pythagorean fuzzy environment.

The remainder of this paper is organized as follows: In Section 2, the concepts of IFS and PFS are briefly reviewed. In Section 3, the fuzzy measure and Choquet integral are retrospected. Moreover, based on operational laws and fuzzy measure, the Pythagorean fuzzy Choquet integral operators are proposed, and some of their properties are investigated in detail. In Section 4, we explore two approaches to MAGDM with attributes involving dependent and independent by the PFCIA operator and MABAC in Pythagorean fuzzy environment. In Section 5, two examples are given to illustrate the concrete applications of the methods and to demonstrate their feasibility and practicality. The paper is concluded in Section 6.

2. PRELIMINARIES

DEFINITION 1.[1] *Let X be a universe of discourse. An IFS I in X is given by*

$$I = \{\langle x_i, \mu_I(x_i), \nu_I(x_i)\rangle \mid x_i \in X\}, \tag{1}$$

where $\mu_I : X \to [0,1]$ denotes the degree of membership and $\nu_I : X \to [0,1]$ denotes the degree of nonmembership of the element $x_i \in X$ to the set I, respectively, with the condition that $0 \le \mu_I(x_i) + \nu_I(x_i) \le 1$. The degree of indeterminacy $\pi_I(x_i) = 1 - \mu_I(x_i) - \nu_I(x_i)$.

DEFINITION 2.[14] *Let X be a universe of discourse. A PFS P in X is given by*

$$P = \{\langle x_i, \mu_P(x_i), \nu_P(x_i)\rangle \mid x_i \in X\}, \tag{2}$$

where $\mu_P : X \to [0,1]$ denotes the degree of membership and $\nu_P : X \to [0,1]$ denotes the degree of nonmembership of the element $x_i \in X$ to the set P, respectively, with the condition that $0 \le (\mu_P(x_i))^2 + (\nu_P(x_i))^2 \le 1$. The degree of indeterminacy $\pi_P(x_i) = \sqrt{1 - (\mu_P(x_i))^2 - (\nu_P(x_i))^2}$. For convenience, Zhang and Xu[17] called $p(x_i) = (\mu_P(x_i), \nu_P(x_i))$ a PFN.

Based on the above definition, Zhang and Xu[17] defined the distance between $p(x_i)$ and $p(x_j)$ as follows:

$$d(p(x_i), p(x_j)) = \frac{1}{2}(|(\mu_P(x_i))^2 - (\mu_P(x_j))^2| + |(\nu_P(x_i))^2 - (\nu_P(x_j))^2| + |(\pi_P(x_i))^2 - (\pi_P(x_j))^2|). \tag{3}$$

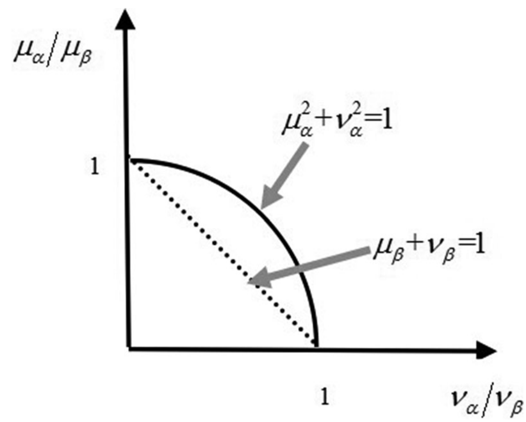

Figure 1. Comparison of spaces of the PFNs and IFNs.

The main difference between PFNs and IFNs is their corresponding constraint conditions which are shown in Figure 1.[15]

DEFINITION 3.[17] *For any PFN $p(x_i) = (\mu_P(x_i), \nu_P(x_i))$, the score function of $p(x_i)$ is defined as follows:*

$$s(p(x_i)) = (\mu_P(x_i))^2 - (\nu_P(x_i))^2, \tag{4}$$

where $s(p(x_i)) \in [-1, 1]$.

DEFINITION 4.[20] *For any PFN $p(x_i) = (\mu_P(x_i), \nu_P(x_i))$, the accuracy function of $p(x_i)$ is defined as follows:*

$$a(p(x_i)) = (\mu_P(x_i))^2 + (\nu_P(x_i))^2, \tag{5}$$

where $a(p(x_i)) \in [0, 1]$.

For any two PFNs $p(x_i)$, $p(x_j)$,

(1) if $s(p(x_i)) > s(p(x_j))$, then $p(x_i) \succ p(x_j)$;
(2) if $s(p(x_i)) = s(p(x_j))$, then
 (a) if $a(p(x_i)) > a(p(x_j))$, then $p(x_i) \succ p(x_j)$;
 (b) if $a(p(x_i)) = a(p(x_j))$, then $p(x_i) = p(x_j)$.

DEFINITION 5.[17,20] *Let $p(x_i) = (\mu_P(x_i), \nu_P(x_i))$ and $p(x_j) = (\mu_P(x_j), \nu_P(x_j))$, and $\lambda > 0$, then the operations can be defined as follows:*

(1) $\lambda p(x_i) = (\sqrt{1 - (1 - (\mu_P(x_i))^2)^\lambda}, (\nu_P(x_i))^\lambda)$;
(2) $p(x_i)^\lambda = ((\mu_P(x_i))^\lambda, \sqrt{1 - (1 - (\nu_P(x_i))^2)^\lambda})$;
(3) $p(x_i) \oplus p(x_j) = (\sqrt{\mu_P^2(x_i) + \mu_P^2(x_j) - \mu_P^2(x_i)\mu_P^2(x_j)}, \nu_P(x_i)\nu_P(x_j))$;

(4) $p(x_i) \otimes p(x_j) = (\mu_P(x_i)\mu_P(x_j), \sqrt{\nu_P^2(x_i) + \nu_P^2(x_j) - \nu_P^2(x_i)\nu_P^2(x_j)})$;

(5) $p(x_i) \ominus p(x_j) = (\sqrt{\frac{\mu_P^2(x_i)-\mu_P^2(x_j)}{1-\mu_P^2(x_j)}}, \frac{\nu_P(x_i)}{\nu_P(x_j)})$, *if* $\mu_P(x_i) \geq \mu_P(x_j), \nu_P(x_i) \leq \min\{\nu_P(x_j), \frac{\nu_P(x_j)\pi_P(x_i)}{\pi_P(x_j)}\}$;

(6) $p(x_i) \oslash p(x_j) = (\frac{\mu_P(x_i)}{\mu_P(x_j)}, \sqrt{\frac{\nu_P^2(x_i)-\nu_P^2(x_j)}{1-\nu_P^2(x_j)}})$, *if* $\nu_P(x_i) \geq \nu_P(x_j), \mu_P(x_i) \leq \min\{\mu_P(x_j), \frac{\mu_P(x_j)\pi_P(x_i)}{\pi_P(x_j)}\}$.

THEOREM 1.[17,20] *Let $p(x_i)$ and $p(x_j)$ be two PFNs, and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then*

(1) $p(x_i) \oplus p(x_j) = p(x_j) \oplus p(x_i)$;
(2) $p(x_i) \otimes p(x_j) = p(x_j) \otimes p(x_i)$;
(3) $\lambda(p(x_i) \oplus p(x_j)) = \lambda p(x_i) \oplus \lambda p(x_j)$;
(4) $\lambda_1 p(x_i) \oplus \lambda_2 p(x_i) = (\lambda_1 + \lambda_2)p(x_i)$;
(5) $(p(x_i) \otimes p(x_j))^\lambda = p(x_i)^\lambda \otimes p(x_j)^\lambda$;
(6) $p(x_i)^{\lambda_1} \otimes p(x_i)^{\lambda_2} = p(x_i)^{(\lambda_1+\lambda_2)}$;
(7) $\lambda(p(x_i) \ominus p(x_j)) = \lambda p(x_i) \ominus \lambda p(x_j)$, *if* $\mu_P(x_i) \geq \mu_P(x_j)$, $\nu_P(x_i) \leq min\ \{\nu_P(x_j), \frac{\nu_P(x_j)\pi_P(x_i)}{\pi_P(x_j)}\}$;
(8) $(p(x_i) \oslash p(x_j))^\lambda = p(x_i)^\lambda \oslash p(x_j)^\lambda$, *if* $\nu_P(x_i) \geq \nu_P(x_j)$, $\mu_P(x_i) \leq min\ \{\mu_P(x_j), \frac{\mu_P(x_j)\pi_P(x_i)}{\pi_P(x_j)}\}$;
(9) $\lambda_1 p(x_i) \ominus \lambda_2 p(x_i) = (\lambda_1 - \lambda_2)p(x_i)$, *if* $\lambda_1 \geq \lambda_2$;
(10) $p(x_i)^{\lambda_1} \oslash p(x_i)^{\lambda_2} = p(x_i)^{(\lambda_1-\lambda_2)}$, *if* $\lambda_1 \geq \lambda_2$.

3. PYTHAGOREAN FUZZY CHOQUET INTEGRAL OPERATORS

3.1. Fuzzy Measure and Choquet Integral Operator

Fuzzy measure (a nonadditive measure), initiated by Sugneo,[32] makes a monotonic property instead of an additive property. For MADM problems, it does not need assumption that criteria or preferences are independent of one another and is used as a powerful tool for modeling interaction phenomena in decision making. In the Choquet integral model,[36,37] where criteria can be dependent, a fuzzy measure is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria. In this subsection, definitions of fuzzy measure, λ-fuzzy measure, discrete Choquet integral, and Pythagorean fuzzy Choquet integral operators are presented as follows:

DEFINITION 6.[32] *A fuzzy measure on X is a set function $\mu : P(X) \to [0, 1]$, satisfying the following conditions:*

(1) $\mu(\phi) = 0, \mu(X) = 1$ *(boundary conditions);*
(2) *If $A, B \in X$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$(monotonicity).*

Even though it is necessary to add the axiom of continuity when X is infinite, it is enough to consider a finite universal set in actual practice. $\mu(\{x_1, x_2, \ldots, x_n\})$ can be considered as the grade of subjective importance of decision attribute set $\{x_1, x_2, \ldots, x_n\}$. Thus, with the separate weights of attributes, weights of any *combination of attributes can also be defined. Some remarks about any pair of attribute sets $A, B \in X$ with the condition $A \cap B \in \phi$ are as follows:*

(i) A and B are independent (without interaction) if $\mu(A \cup B) = \mu(A) + \mu(B)$. It is called an additive measure.
(ii) Positive interaction between A and B is exhibited if $\mu(A \cup B) > \mu(A) + \mu(B)$. It is called a superadditive measure.
(iii) Negative interaction between A and B is exhibited if $\mu(A \cup B) < \mu(A) + \mu(B)$. It is called a subadditive measure.

Since it is difficult to determine the fuzzy measure according to Definition 5, therefore, to confirm a fuzzy measure in MAGDM problems, Sugeno[32] *presented the following λ-fuzzy measure:*

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B), \quad \lambda \in [-1, \infty), A \cap B = \phi. \quad (6)$$

The parameter λ determines interaction between the attributes. In Equation 6*, if $\lambda = 0$, λ-fuzzy measure reduces to simply an additive measure. And for negative and positive λ, the λ-fuzzy measure reduces to subadditive and superadditive measures, respectively. Meanwhile, if all the elements in X are independent, and we have*

$$\mu(A) = \sum_{x_i \in A} \mu(\{x_i\}). \quad (7)$$

DEFINITION 7.[47] *Let f be a positive real-valued function on X and μ be a fuzzy measure on X. The discrete Choquet integral of f with respect to μ is defined by*

$$C_\mu(f) = \sum_{i=1}^{n} f_{\sigma(i)}[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})], \quad (8)$$

where $\sigma(i)$ indicates a permutation on X such that $f_{\sigma(1)} \geq f_{\sigma(2)} \geq \cdots \geq f_{\sigma(n)}$, $A_{\sigma(i)} = \{1, 2, \ldots, i\}$, $A_{\sigma(0)} = \phi$.

It is seen that the discrete Choquet integral is a linear expression up to a reordering of the elements. Moreover, it identifies with the weighted mean (discrete Lebesgue integral) as soon as the fuzzy measure is additive. And in some condition, the Choquet integral operator coincides with the OWA (OWA) operator.[48]

3.2. Pythagorean Fuzzy Choquet Integral Operators

In this subsection, we define Pythagorean fuzzy Choquet integral operators to take into account the interaction phenomena between the attribute represented by the fuzzy measure.

DEFINITION 8.[19] *Let $p(x_i) = (\mu_P(x_i), \nu_P(x_i))(i = 1, 2, \partial, n)$ be a collection of PFNs on X and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $p(x_i)(i = 1, 2, \ldots, n)$,* *with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, *then a PFWA operator is a mapping PFWA:* $P^n \to P$, *where*

$$\mathrm{PFWA}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\sum_{i=1}^{n} w_i \mu_p(x_i), \sum_{i=1}^{n} w_i \nu_p(x_i)\right). \quad (9)$$

DEFINITION 9.[19] *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs and* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* $p(x_i)(i = 1, 2, \ldots, n)$, *with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, *then a PFWG operator is a mapping PFWG:* $P^n \to P$, *where*

$$\mathrm{PFWG}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\prod_{i=1}^{n} (\mu_p(x_i))^{w_i}, \prod_{i=1}^{n} (\nu_p(x_i))^{w_i}\right). \quad (10)$$

DEFINITION 10. *Let* $p(x_i) = (\mu_P(x_i), \nu_P(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on X and* $w = (w_1, w_2, \partial, w_n)^T$ *be the weight vector of* $p(x_{\sigma(i)})(i = 1, 2, \ldots, n)$, *with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$, *then a PFOWA operator is a mapping PFOWA:* $P^n \to P$, *where*

$$\mathrm{PFOWA}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\sum_{i=1}^{n} w_i \mu_p(x_{\sigma(i)}), \sum_{i=1}^{n} w_i \nu_p(x_{\sigma(i)})\right). \quad (11)$$

DEFINITION 11. *Let* $p(x_i) = (\mu_P(x_i), \nu_P(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on X and* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* $p(x_{\sigma(i)})(i = 1, 2, \ldots, n)$, *with* $w_i \in [0, 1]$, $\sum_{i=1}^{n} w_i = 1$, $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$, *then a Pythagorean fuzzy ordered weighted geometric (PFOWG) operator is a mapping PFOWG:* $P^n \to P$, *where*

$$\mathrm{PFOWG}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\prod_{i=1}^{n} (\mu_p(x_{\sigma(i)}))^{w_i}, \prod_{i=1}^{n} (\nu_p(x_{\sigma(i)}))^{w_i}\right). \quad (12)$$

Based on Definitions 7–10, we first give the definition of the Pythagorean fuzzy Choquet integral operators as follows:

DEFINITION 12. *Let* $p(x_i) = (\mu_P(x_i), \nu_P(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on X,* μ *be a fuzzy measure on X, then a Pythagorean fuzzy Choquet integral average (PFCIA) operator is a mapping PFCIA:* $P^n \to P$, *where*

$$\begin{aligned}\mathrm{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) = \Bigg(&\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)}),\\ &\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\Bigg), \quad (13)\end{aligned}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$ *such that* $p(x_{\sigma(1)}) \geq p(x_{\sigma(2)}) \geq \ldots \geq p(x_{\sigma(n)})$, $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ *for* $k \geq 1$, *and* $A_{\sigma(0)} = \phi$.

Now we consider four special cases of the PFCIA operator:

(1) If Equation 7 *holds, then* $\mu(\{x_{\sigma(i)}\}) = \mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$, *i.e., Equation* 13 *reduces to the PFWA operator shown in Equation* 9.
(2) If $\mu(A) = \sum_{i=1}^{|A|} w_i, \forall A \in X$, *where* $|A|$ *is the number of the elements in the set A,* $w_i = \mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$, *where* $w = (w_1, w_2, \ldots, w_n)^T$, *with* $\sum_{i=1}^{n} w_i = 1$, *i.e., Equation* 13 *reduces to the PFOWA operator as shown in Equation* 11.
(3) If $\mu(A) = 1, \forall A \in X$, *then* $\mathrm{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) = \max\{p(x_1), p(x_2), \ldots, p(x_n)\} = p(x_{\sigma(1)})$.
(4) If $\mu(A) = 0, \forall A \in X$, *then* $\mathrm{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) = \min\{p(x_1), p(x_2), \ldots, p(x_n)\} = p(x_{\sigma(n)})$.

DEFINITION 13. *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on X,* μ *be a fuzzy measure on X, then a Pythagorean fuzzy Choquet integral geometric (PFCIG) operator is a mapping PFCIG:* $P^n \to P$, *where*

$$\mathrm{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\prod_{i=1}^{n} (\mu_p(x_i))^{\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})}, \prod_{i=1}^{n} (\nu_p(x_i))^{\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})}\right), \tag{14}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$ *such that* $p(x_{\sigma(1)}) \geq p(x_{\sigma(2)}) \geq \cdots \geq p(x_{\sigma(n)})$, $A_{\sigma(k)} = \{x_{\sigma(j)} | j \leq k\}$ *for* $k \geq 1$, *and* $A_{\sigma(0)} = \phi$.

Now we consider four special cases of the PFCIG operator:

(1) If Equation 7 *holds, then* $\mu(\{x_{\sigma(i)}\}) = \mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$, *i.e., Equation* 14 *reduces to the PFWG operator shown in Equation* 10.
(2) If $\mu(A) = \sum_{i=1}^{|A|} w_i, \forall A \in X$, *where* $|A|$ *is the number of the elements in the set A,* $w_i = \mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})(i = 1, 2, \ldots, n)$, *where* $w = (w_1, w_2, \ldots, w_n)^T$, *with* $\sum_{i=1}^{n} w_i = 1$, *i.e., Equation* 14 *reduces to the PFOWG operator shown in Equation* 12.
(3) If $\mu(A) = 1, \forall A \in X$, *then* $\mathrm{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) = \max\{p(x_1), p(x_2), \ldots, p(x_n)\} = p(x_{\sigma(1)})$.
(4) If $\mu(A) = 0, \forall A \in X$, *then* $\mathrm{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) = \min\{p(x_1), p(x_2), \ldots, p(x_n)\} = p(x_{\sigma(n)})$.

LEMMA 1. *Let* μ *be a fuzzy measure,* $A \in X$, *and* $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, n\}$, *then*

$$\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})) = 1. \tag{15}$$

Proof. It is obvious that $\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))$

$$= \mu(A_{\sigma(1)}) - \mu(A_{\sigma(0)}) + \mu(A_{\sigma(2)}) - \mu(A_{\sigma(1)}) + \cdots + \mu(A_{\sigma(n)}) - \mu(A_{\sigma(n-1)})$$
$$= \mu(A_{\sigma(n)}) - \mu(A_{\sigma(0)}) = \mu(X) - \mu(\phi) = 1 - 0 = 1.$$

□

LEMMA 2. [31] *Let* $0 \le a \le 1$, $0 \le b \le 1$, *and* $0 \le x \le 1$, *then*

$$0 \le ax + b(1-x) \le 1. \tag{16}$$

LEMMA 3. [31] *Let* $0 \le x \le 1$, $\lambda \ge 0$, *and* $f(x) = x^{\lambda} + (1-x)^{\lambda}$, *then*

$$f(x) : \begin{cases} 0 < f(x) \le 1 & \text{iff } \lambda \ge 1 \\ f(x) \ge 1 & \text{iff } 0 \le \lambda \le 1 \end{cases}$$

THEOREM 2. *(Idempotency) Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on X, and* μ *be a fuzzy measure on X. If all* $p(x_i)(i = 1, 2, \ldots, n)$ *are equal, i.e., for all* i, $p(x_i) = p(x) = (\mu_p(x), \nu_p(x))$, *then*

(1) $\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) = p(x)$;
(2) (2) $\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) = p(x)$.

Proof: In the following, we shall prove (1), and (2) can be proved analogously.
(1) According to Definition 11, for $\forall i$, $p(x_i) = p(x)$, *then*

$$\begin{aligned} &\text{PFCIA}\ (p(x_1), p(x_2), \ldots, p(x_n)) \\ &= \left(\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)}), \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)}) \right) \\ &= \left(\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x), \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x) \right) \\ &= (\mu_p(x), \nu_p(x)) = p(x) \quad \text{(According to Lemma 1).} \end{aligned}$$

THEOREM 3. *(Monotonicity) Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))$ *and* $q(x_i) = (\mu_q(x_i), \nu_q(x_i))(i = 1, 2, \ldots, n)$ *be two collections of PFNs on X, and* μ *be a fuzzy measure on X. If* $p(x_{\sigma(1)}) \le p(x_{\sigma(2)}) \le \cdots \le p(x_{\sigma(n)})$ *and* $q(x_{\sigma(1)}) \le q(x_{\sigma(2)}) \le \cdots \le q(x_{\sigma(n)})$, *and* $\forall i$, $p(x_{\sigma(i)}) \le q(x_{\sigma(i)})$, *i.e.,* $\mu_p(x_{\sigma(i)}) \le \mu_q(x_{\sigma(i)})$, $\nu_p(x_{\sigma(i)}) \ge \nu_q(x_{\sigma(i)})$, *then*

(1) $\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \le \text{PFCIA}(q(x_1), q(x_2), \ldots, q(x_n))$;
(2) $\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \le \text{PFCIG}(q(x_1), q(x_2), \ldots, q(x_n))$.

Proof. In the following, we shall prove (1), and (2) can be proved analogously.
(1) According to Definition 11, we have

$$\begin{aligned} &\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \\ &= \left(\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)}), \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)}) \right), \end{aligned}$$

$$\text{PFCIA}(q(x_1), q(x_2), \ldots, q(x_n)) = \left(\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)}), \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)}) \right).$$

Also, $\forall i$, $\mu_p(x_{\sigma(i)}) \leq \mu_q(x_{\sigma(i)})$, $\nu_p(x_{\sigma(i)}) \geq \nu_q(x_{\sigma(i)})$, and $\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}) > 0$, we have

$$\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)}) \leq \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)}),$$

$$\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)}) \geq \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)}).$$

According to Equation 4, this proof is completed. □

THEOREM 4. *Let $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ be a collection of PFNs on X, and μ be a fuzzy measure on X, $p^-(x) = (\min_i\{\mu_p(x_i)\}, \max_i\{\nu_p(x_i)\})$, $p^+(x) = (\max_i\{\mu_p(x_i)\}, \min_i\{\nu_p(x_i)\})$, then*

(1) $p^-(x) \leq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \leq p^+(x)$;
(2) $p^-(x) \leq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \leq p^+(x)$.

Proof. In the following, we shall prove (1), and (2) can be proved analogously.
(1) According to Definition 11, we have

$$\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) = \left(\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)}), \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)}) \right).$$

Obviously,

$$\min_i\{\mu_p(x_i)\} \leq \mu_p(x_{\sigma(i)}) \leq \max_i\{\mu_p(x_i)\}, \min_i\{\nu_p(x_i)\} \leq \nu_p(x_{\sigma(i)}) \leq \max_i\{\nu_p(x_i)\}.$$

Moreover, $\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}) > 0$.

Therefore, we have

$$\sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})) \min_i\{\mu_p(x_i)\} \leq \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})$$
$$\leq \sum_{i=1}^{n} (\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})) \max_i\{\mu_p(x_i)\},$$

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\min_i\{\nu_p(x_i)\}\leq\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})$$
$$\leq\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\max_i\{\nu_p(x_i)\}.$$

According to Lemma 1, we have

$$\min_i\{\mu_p(x_i)\}\leq\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})\leq\max_i\{\mu_p(x_i)\},$$
$$\min_i\{\nu_p(x_i)\}\leq\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\leq\max_i\{\nu_p(x_i)\}.$$

According to Equation 4, this proof is completed. □

THEOREM 5. *Let $p(x_i)=(\mu_p(x_i),\nu_p(x_i))(i=1,2,\ldots,n)$ be a collection of PFNs on X, $p(x)=(\mu_p(x),\nu_p(x))$ be a PFN and μ be a fuzzy measure on X, then*

(1) PFCIA$(p(x_1)\oplus p(x), p(x_2)\oplus p(x),\ldots, p(x_n)\oplus p(x))\geq$ PFCIA$(p(x_1)\otimes p(x), p(x_2)\otimes p(x),\ldots, p(x_n)\otimes p(x))$;
(2) PFCIG$(p(x_1)\oplus p(x), p(x_2)\oplus p(x),\ldots, p(x_n)\oplus p(x))\geq$ PFCIG$(p(x_1)\otimes p(x), p(x_2)\otimes p(x),\ldots, p(x_n)\otimes p(x))$;
(3) PFCIA$(p(x_1)\oplus p(x), p(x_2)\oplus p(x),\ldots, p(x_n)\oplus p(x))\geq$ PFCIA $(p(x_1), p(x_2),\ldots, p(x_n))\otimes p(x)$;
(4) PFCIG$(p(x_1)\oplus p(x), p(x_2)\oplus p(x),\ldots,$ PFCIA $\oplus p(x))\geq$ PFCIG$(p(x_1), p(x_2),\ldots, p(x_n))\otimes p(x)$;
(5) PFCIA$(p(x_1), p(x_2),\ldots, p(x_n))\oplus p(x)\geq$ *PFCIA* $(p(x_1)\otimes p(x), p(x_2)\otimes p(x),\ldots, p(x_n)\otimes p(x))$;
(6) PFCIG$(p(x_1), p(x_2),\ldots, p(x_n))\oplus p(x)\geq$ PFCIG$(p(x_1)\otimes p(x), p(x_2)\otimes p(x),\ldots, p(x_n)\otimes p(x))$;
(7) PFCIA$(p(x_1), p(x_2),\ldots, p(x_n))\oplus p(x)\geq$ PFCIA$(p(x_1), p(x_2),\ldots, p(x_n))\otimes p(x)$;
(8) PFCIG$(p(x_1), p(x_2),\ldots, p(x_n))\oplus p(x)\geq$ PFCIG$(p(x_1), p(x_2),\ldots, p(x_n))\otimes p(x)$.

Proof. In the following, we shall prove (1), and (2)–(8) can be proved analogously.

(1) For any $p(x_{\sigma(i)})=(\mu_p(x_{\sigma(i)}),\nu_p(x_{\sigma(i)})(i=1,2,\ldots,n)$ and $p(x)=(\mu_p(x),\nu_p(x)$, we can have

$$\sqrt{(\mu_p(x_{\sigma(i)}))^2+(\mu_p(x))^2-(\mu_p(x_{\sigma(i)}))^2(\mu_p(x))^2}$$
$$\geq\sqrt{2(\mu_p(x_{\sigma(i)}))^2(\mu_p(x))^2-(\mu_p(x_{\sigma(i)}))^2(\mu_p(x))^2}=\mu_p(x_{\sigma(i)})\mu_p(x),$$
$$\sqrt{(\nu_p(x_{\sigma(i)}))^2+(\nu_p(x))^2-(\nu_p(x_{\sigma(i)}))^2(\nu_p(x))^2}$$
$$\geq\sqrt{2(\nu_p(x_{\sigma(i)}))^2(\nu_p(x))^2-(\nu_p(x_{\sigma(i)}))^2(\nu_p(x))^2}=\nu_p(x_{\sigma(i)})\nu_p(x).$$

Moreover, $\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}) > 0$; consequently,

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^2 + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^2(\mu_p(x))^2}$$

$$\geq \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})\mu_p(x),$$

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\nu_p(x_{\sigma(i)}))^2 + (\nu_p(x))^2 - (\nu_p(x_{\sigma(i)}))^2(\nu_p(x))^2}$$

$$\geq \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_p(x).$$

According to Definition 11, we have

$$PFCIA \int (p(x_1) \oplus p(x), p(x_2) \oplus p(x), \ldots, p(x_n) \oplus p(x))$$

$$= \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^2 + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^2(\mu_p(x))^2},\right.$$

$$\left.\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_p(x)\right),$$

$$PFCIA\,(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$$

$$= \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})\mu_p(x), \sum_{i=1}^{n}\mu(A_{\sigma(i)})\right.$$

$$\left. - \mu(A_{\sigma(i-1)})\sqrt{(\nu_p(x_{\sigma(i)}))^2 + (\nu_p(x))^2 - (\nu_p(x_{\sigma(i)}))^2(\nu_p(x))^2}\right).$$

According to Equation 4, this proof is completed. □

THEOREM 6. *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on* X, $p(x) = (\mu_p(x), \nu_p(x))$ *be a PFN and* μ *be a fuzzy measure on* X, $\lambda \geq 0$, *then*

(1) $\text{PFCIA}(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x)) \geq \text{PFCIA}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x))$;
(2) $\text{PFCIG}(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x)) \geq \text{PFCIG}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x))$;
(3) $\text{PFCIA}(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \geq \text{PFCIA}(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$;

(4) PFCIG$(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \geq$ PFCIG$(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$;

(5) PFCIA$(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \geq$ PFCIA$(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x))$, *iff* $\lambda \geq 1$;
PFCIA$(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \leq$ PFCIA$(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x))$, *iff* $0 \leq \lambda \leq 1$;

(6) PFCIG$(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \geq$ PFCIG$(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x))$, *iff* $\lambda \geq 1$;
PFCIG$(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x)) \leq$ PFCIG$(p(x_1)^\lambda \oplus p(x), p(x_2)^\lambda \oplus p(x), \ldots, p(x_n)^\lambda \oplus p(x))$, *iff* $0 \leq \lambda \leq 1$;

(7) PFCIA$(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \geq$ PFCIA$(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$, *iff* $\lambda \geq 1$;
PFCIA$(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \leq$ PFCIA$(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$, *iff* $0 \leq \lambda \leq 1$;

(8) PFCIG$(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \geq$ PFCIG$(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$, *iff* $\lambda \geq 1$;
PFCIG$(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \leq$ PFCIG$(p(x_1)^\lambda \otimes p(x), p(x_2)^\lambda \otimes p(x), \ldots, p(x_n)^\lambda \otimes p(x))$, *iff* $0 \leq \lambda \leq 1$.

Proof. In the following, we shall prove (1), (5), and (2)–(4) with (6)–(8) can be proved analogously.

(1) For any $p(x_{\sigma(i)}) = (\mu_p(x_{\sigma(i)}), \nu_p(x_{\sigma(i)}))(i = 1, 2, \ldots, n)$ and $p(x) = (\mu_p(x), \nu_p(x)$, let

$$\begin{aligned} f(\mu_p(x_{\sigma(i)})) &= (\mu_p(x_{\sigma(i)}))^{2\lambda} + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2 \\ &\quad - [(\mu_p(x))^2 - (\mu_p(x))^2(1 - \mu_p(x_{\sigma(i)}))^\lambda] \\ &= (\mu_p(x_{\sigma(i)}))^{2\lambda}(1 - (\mu_p(x))^2) + (1 - \mu_p(x_{\sigma(i)}))^\lambda(\mu_p(x))^2. \end{aligned}$$

Since $0 \leq (\mu_p(x_{\sigma(i)}))^{2\lambda} \leq 1$, $0 \leq 1 - (\mu_p(x))^2 \leq 1$, $0 \leq (1 - \mu_p(x_{\sigma(i)}))^\lambda \leq 1$, $0 \leq (\mu_p(x))^2 \leq 1$, and according to Lemma 2, we can have $f(\mu_p(x_{\sigma(i)})) \geq 0$, i.e.,

$$\begin{aligned} &(\mu_p(x_{\sigma(i)}))^{2\lambda} + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2 \\ &\geq (\mu_p(x))^2 - (\mu_p(x))^2(1 - \mu_p(x_{\sigma(i)}))^\lambda. \end{aligned}$$

Because $\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}) > 0$, therefore,

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^{2\lambda} + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2}$$

$$\geq \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x))^2 - (\mu_p(x))^2(1 - \mu_p(x_{\sigma(i)}))^\lambda}.$$

Similarly,

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\nu_p(x_{\sigma(i)}))^{2\lambda}+(\nu_p(x))^2-(\nu_p(x_{\sigma(i)}))^{2\lambda}(\nu_p(x))^2}$$

$$\geq \sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\nu_p(x))^2-(\nu_p(x))^2(1-\nu_p(x_{\sigma(i)}))^{\lambda}}.$$

According to Definition 11, we have

$$\text{PFCIA}(p(x_1)^{\lambda}\oplus p(x), p(x_2)^{\lambda}\oplus p(x), \ldots, p(x_n)^{\lambda}\oplus p(x)) =$$

$$\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^{2\lambda}+(\mu_p(x))^2-(\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2},\right.$$

$$\left.\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\nu_p(x))^2-(\nu_p(x))^2(1-\nu_p(x_{\sigma(i)}))^{\lambda}}\right),$$

$$\text{PFCIA}(\lambda p(x_1)\otimes p(x), \lambda p(x_2)\otimes p(x), \ldots, \lambda p(x_n)\otimes p(x))$$

$$=\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x))^2-(\mu_p(x))^2(1-\mu_p(x_{\sigma(i)}))^{\lambda}},\right.$$

$$\left.\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\sqrt{(\nu_p(x_{\sigma(i)}))^{2\lambda}+(\nu_p(x))^2-(\nu_p(x_{\sigma(i)}))^{2\lambda}(\nu_p(x))^2}\right).$$

According to Equation 4, this proof is completed. □

(5) For any $p(x_{\sigma(i)}) = (\mu_p(x_{\sigma(i)}), \nu_p(x_{\sigma(i)}))(i = 1, 2, \ldots, n)$ and $p(x) = (\mu_p(x), \nu_p(x)$, let

$$\begin{aligned} f(\mu_p(x_{\sigma(i)})) &= 1-(1-(\mu_p(x))^2)(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda}-[(\mu_p(x_{\sigma(i)}))^{2\lambda}+(\mu_p(x))^2 \\ &\quad -(\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2] \\ &= (1-(\mu_p(x))^2)[1-((\mu_p(x_{\sigma(i)}))^2)^{\lambda}-(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda}], \\ g(\nu_p(x_{\sigma(i)})) &= (\nu_p(x_{\sigma(i)})^{2\lambda}-[1-(1-(\nu_p(x_{\sigma(i)})^2)^{\lambda}] \\ &= (1-(\nu_p(x_{\sigma(i)})^2)^{\lambda}+((\nu_p(x_{\sigma(i)})^2)^{\lambda}-1. \end{aligned}$$

We denote $s(\mu_p(x_{\sigma(i)})) = ((\mu_p(x_{\sigma(i)}))^2)^{\lambda}-(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda}$, $t(\mu_p(x_{\sigma(i)})) = ((\nu_p(x_{\sigma(i)}))^2)^{\lambda}-(1-(\nu_p(x_{\sigma(i)}))^2)^{\lambda}$.

According to Lemma 3, we can have

$$\text{if } 0 \leq \lambda \leq 1, \text{ then } s(\mu_p(x_{\sigma(i)})) \geq 1, t(\mu_p(x_{\sigma(i)})) \geq 1$$
$$\text{and if } \lambda \geq 1, \text{ then } 0 \leq s(\mu_p(x_{\sigma(i)})) \leq 1, 0 \leq t(\mu_p(x_{\sigma(i)})) \leq 1.$$

Furthermore, we can have

$$\begin{aligned}
&f(\mu_p(x_{\sigma(i)})) \geq 0, t(\mu_p(x_{\sigma(i)})) \leq 0 \text{ when } \lambda \geq 1,\\
&\text{and } f(\mu_p(x_{\sigma(i)})) \leq 0, t(\mu_p(x_{\sigma(i)})) \geq 0 \text{ when } 0 \leq \lambda \leq 1, \text{ i.e.,}\\
&1-(1-(\mu_p(x))^2)(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda} \geq (\mu_p(x_{\sigma(i)}))^{2\lambda}\\
&\quad + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2,\\
&\nu_p(x)(\nu_p(x_{\sigma(i)})^{2\lambda} \leq \nu_p(x)(1-(1-(\nu_p(x_{\sigma(i)})^2)^{\lambda}) \text{ when } \lambda \geq 1, \text{ and}\\
&1-(1-(\mu_p(x))^2)(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda} \leq (\mu_p(x_{\sigma(i)}))^{2\lambda}\\
&\quad + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2,\\
&\nu_p(x)(\nu_p(x_{\sigma(i)})^{2\lambda} \geq \nu_p(x)(1-(1-(\nu_p(x_{\sigma(i)})^2)^{\lambda}) \text{ when } 0 \leq \lambda \leq 1.
\end{aligned}$$

Moreover, $\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}) > 0$, consequently,

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(1-(1-(\mu_p(x))^2)(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda})}$$

$$\geq \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^{2\lambda} + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2)},$$

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x)(\nu_p(x_{\sigma(i)}))^{\lambda} \leq \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x)$$
$$\sqrt{(1-(1-(\nu_p(x_{\sigma(i)})^2)^{\lambda}))}.$$

According to Definition 11, we have

$$\begin{aligned}
&\text{PFCIA}(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x))\\
&= \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{1-(1-(\mu_p(x))^2)(1-(\mu_p(x_{\sigma(i)}))^2)^{\lambda}},\right.\\
&\left.\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x)(\nu_p(x_{\sigma(i)}))^{\lambda}\right),
\end{aligned}$$

$$\text{PFCIA}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x))$$
$$= \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)})) \sqrt{(\mu_p(x_{\sigma(i)}))^{2\lambda} + (\mu_p(x))^2 - (\mu_p(x_{\sigma(i)}))^{2\lambda}(\mu_p(x))^2}, \sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x)\sqrt{1-(1-(\nu_p(x_{\sigma(i)})^2)^{\lambda})}\right).$$

According to Equation 4, this proof is completed. □

THEOREM 7. *Let $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ be a collection of PFNs on X, $p(x) = (\mu_p(x), \nu_p(x))$ be a PFN and μ be a fuzzy measure on X, $\lambda \geq 0$, then*

(1) $\lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \geq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$;
(2) $\lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \geq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$;
(3) $\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \geq \lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$;
(4) $\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \geq \lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$;
(5) $\lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \geq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$, *iff* $\lambda \geq 1$;
$\lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \leq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$, *iff* $0 \leq \lambda \leq 1$;
(6) $\lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \geq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$, *iff* $\lambda \geq 1$;
$\lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \leq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$, *iff* $0 \leq \lambda \leq 1$;
(7) $\lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$, *iff* $\lambda \geq 1$;
$\lambda\text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq \text{PFICA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$;
(8) $\lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$, *iff* $\lambda \geq 1$;
$\lambda\text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq \text{PFICG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$.

Proof. The proof is similar to Theorem 6. □

THEOREM 8. *Let $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ be a collection of PFNs on X, $p(x) = (\mu_p(x), \nu_p(x))$ be a PFN and μ be a fuzzy measure on X, $\lambda \geq 0$, then*

(1) $\text{PFCIA}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \leq \text{PFCIA}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \leq \text{PFCIA}(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x))$;
(2) $\text{PFCIG}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \leq \text{PFCIG}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \leq \text{PFCIG}(\lambda p(x_1) \oplus p(x), \lambda p(x_2) \oplus p(x), \ldots, \lambda p(x_n) \oplus p(x))$;
(3) $\text{PFCIA}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \leq \text{PFCIA}(p(x_1)^{\lambda}, p(x_2)^{\lambda}, \ldots, p(x_n)^{\lambda}) \leq \text{PFCIA}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x))$;

(4) $\text{PFCIG}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \le \text{PFCIG}(p(x_1)^{\lambda}, p(x_2)^{\lambda}, \ldots, p(x_n)^{\lambda}) \le \text{PFCIG}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x))$;

(5) $\lambda\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \le \lambda\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \le \lambda\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$;

(6) $\lambda\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \le \lambda\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \le \lambda\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$;

(7) $\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \le \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \le \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$;

(8) $\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \le \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \le \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x)$.

Proof. The proof is similar to Theorem 6. □

THEOREM 9. *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on* X, $p(x) = (\mu_p(x), \nu_p(x))$ *be a PFN and* μ *be a fuzzy measure on* X, $\lambda \ge 0$, *then*

(1) $\text{PFCIA}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \le \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIA}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \ge \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))$, *iff* $\lambda \ge 1$;

(2) $\text{PFCIG}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \le \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIG}(\lambda p(x_1), \lambda p(x_2), \ldots, \lambda p(x_n)) \ge \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))$, *iff* $\lambda \ge 1$;

(3) $\text{PFCIA}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \ge \text{PFCIA}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIA}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \le \text{PFCIA}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $\lambda \ge 1$;

(4) $\text{PFCIG}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \ge \text{PFCIG}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIG}(p(x_1)^{\lambda} \otimes p(x), p(x_2)^{\lambda} \otimes p(x), \ldots, p(x_n)^{\lambda} \otimes p(x)) \le \text{PFCIG}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $\lambda \ge 1$;

(5) $\text{PFCIA}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x)) \ge \text{PFCIA}(p(x_1) \oplus p(x), p(x_2) \oplus p(x), \ldots, p(x_n) \oplus p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIA}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x)) \le \text{PFCIA}(p(x_1) \oplus p(x), p(x_2) \oplus p(x), \ldots, p(x_n) \oplus p(x))$, *iff* $\lambda \ge 1$;

(6) $(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x)) \ge \text{PFCIG}(p(x_1) \oplus p(x), p(x_2) \oplus p(x), \ldots, p(x_n) \oplus p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIG}(p(x_1)^{\lambda} \oplus p(x), p(x_2)^{\lambda} \oplus p(x), \ldots, p(x_n)^{\lambda} \oplus p(x)) \le \text{PFCIG}(p(x_1) \oplus p(x), p(x_2) \oplus p(x), \ldots, p(x_n) \oplus p(x))$, *iff* $\lambda \ge 1$;

(7) $(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \le \text{PFCIA}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIA}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \ge \text{PFCIA}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $\lambda \ge 1$;

(8) $\text{PFCIG}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \le \text{PFCIG}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $0 \le \lambda \le 1$; $\text{PFCIG}(\lambda p(x_1) \otimes p(x), \lambda p(x_2) \otimes p(x), \ldots, \lambda p(x_n) \otimes p(x)) \ge \text{PFCIG}(p(x_1) \otimes p(x), p(x_2) \otimes p(x), \ldots, p(x_n) \otimes p(x))$, *iff* $\lambda \ge 1$.

Proof. It is trial. □

THEOREM 10. *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ *be a collection of PFNs on* X, $p(x) = (\mu_p(x), \nu_p(x))$ *be a PFN and* μ *be a fuzzy measure on* X, $\lambda \ge 0$, *then*

(1) $\lambda\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \le \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $0 \le \lambda \le 1$; $\lambda\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \ge \text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $\lambda \ge 1$;

(2) $\lambda\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \le \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $0 \le \lambda \le 1$; $\lambda\text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x) \ge \text{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $\lambda \ge 1$;

(3) λPFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \leq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$; λPFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $\lambda \geq 1$;

(4) λPFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \leq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$; λ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x) \geq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $\lambda \geq 1$;

(5) PFCIA$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \geq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $0 \leq \lambda \leq 1$; PFCIA$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \leq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $\lambda \geq 1$;

(6) PFCIG$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \geq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $0 \leq \lambda \leq 1$; PFCIG$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \oplus p(x) \leq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus p(x)$, *iff* $\lambda \geq 1$;

(7) PFCIA$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \geq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$; PFCIA$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \leq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $\lambda \geq 1$;

(8) (8) PFCIG$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \geq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $0 \leq \lambda \leq 1$; PFCIG$(p(x_1), p(x_2), \ldots, p(x_n))^{\lambda} \otimes p(x) \leq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes p(x)$, *iff* $\lambda \geq 1$.

Proof. It is trial. □

THEOREM 11. *Let* $p(x_i) = (\mu_p(x_i), \nu_p(x_i))$ *and* $q(x_i) = (\mu_q(x_i), \nu_q(x_i))(i = 1, 2, \ldots, n)$ *be two collections of PFNs on X, and* μ *be a fuzzy measure on X, then*

(1) PFCIA$(p(x_1) \oplus q(x_1), p(x_2) \oplus q(x_2), \ldots, p(x_n) \oplus q(x_n))$
$\geq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes$ PFCIA$(q(x_1), q(x_2), \ldots, q(x_n))$;

(2) PFCIG$(p(x_1) \oplus q(x_1), p(x_2) \oplus q(x_2), \ldots, p(x_n) \oplus q(x_n))$
$\geq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes$ PFCIG$(q(x_1), q(x_2), \ldots, q(x_n))$;

(3) PFCIA$(p(x_1) \oplus q(x_1), p(x_2) \oplus q(x_2), \ldots, p(x_n) \oplus q(x_n)) \geq$ PFCIA$(p(x_1) \otimes q(x_1), p(x_2) \otimes q(x_2), \ldots, p(x_n) \otimes q(x_n))$;

(4) PFCIG$(p(x_1) \oplus q(x_1), p(x_2) \oplus q(x_2), \ldots, p(x_n) \oplus q(x_n)) \geq$ PFCIG$(p(x_1) \otimes q(x_1), p(x_2) \otimes q(x_2), \ldots, p(x_n) \otimes q(x_n))$;

(5) PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus$ PFCIA$(q(x_1), q(x_2), \ldots, q(x_n))$
$\geq$ PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes$ PFCIA$(q(x_1), q(x_2), \ldots, q(x_n))$;

(6) PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus$ PFCIG$(q(x_1), q(x_2), \ldots, q(x_n))$
$\geq$ PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \otimes$ PFCIG$(q(x_1), q(x_2), \ldots, q(x_n))$;

(7) PFCIA$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus$ PFCIA$(q(x_1), q(x_2), \ldots, q(x_n))$
$\geq$ PFCIA$(p(x_1) \otimes q(x_1), p(x_2) \otimes q(x_2), \ldots, p(x_n) \otimes q(x_n))$;

(8) PFCIG$(p(x_1), p(x_2), \ldots, p(x_n)) \oplus$ PFCIG$(q(x_1), q(x_2), \ldots, q(x_n))$
$\geq$ PFCIG$(p(x_1) \otimes q(x_1), p(x_2) \otimes q(x_2), \ldots, p(x_n) \otimes q(x_n))$.

Proof. In the following, we shall prove (1), and (2)–(8) can be proved analogously.

(1) For any $p(x_{\sigma(i)}) = (\mu_p(x_{\sigma(i)}), \nu_p(x_{\sigma(i)}))$, $q(x_{\sigma(i)}) = (\mu_q(x_{\sigma(i)}), \nu_q(x_{\sigma(i)}))$ $(i = 1, 2, \ldots, n)$, and according to Definition 11, we have

$$\text{PFCIA}(p(x_1) \oplus q(x_1), p(x_2) \oplus q(x_2), \ldots, p(x_n) \oplus q(x_n))$$

$$= \Bigg(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))$$

$$\times \sqrt{(\mu_p(x_{\sigma(i)}))^2 + (\mu_q(x_{\sigma(i)}))^2 - (\mu_p(x_{\sigma(i)}))^2(\mu_q(x_{\sigma(i)}))^2},$$

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\Bigg),$$

$$\text{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n)) \otimes \text{PFCIA}(q(x_1), \quad q(x_2), \ldots, \quad q(x_n))$$

$$= \left(\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})\right)\right.$$

$$\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)})\right),$$

$$\sqrt{\begin{array}{l}\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2 + \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2 \\ \quad - \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2\end{array}}\Bigg).$$

Since

$$(\mu_p(x_{\sigma(i)}))^2 + (\mu_q(x_{\sigma(i)}))^2 - (\mu_p(x_{\sigma(i)}))^2(\mu_q(x_{\sigma(i)}))^2 - (\mu_p(x_{\sigma(i)}))^2$$

$$\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)})\right)^2$$

$$= (\mu_p(x_{\sigma(i)}))^2\left[1 - \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)})\right)^2\right]$$

$$+ (\mu_q(x_{\sigma(i)}))^2(1 - (\mu_p(x_{\sigma(i)}))^2) \geq 0,$$

so

$$\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\sqrt{(\mu_p(x_{\sigma(i)}))^2 + (\mu_q(x_{\sigma(i)}))^2}$$

$$\geq \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_p(x_{\sigma(i)})\right)\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\mu_q(x_{\sigma(i)})\right).$$

Afterward, we denote

$$
\begin{aligned}
f = &\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2 \\
&+ \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2 \\
&- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2 \\
&\times \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2 \\
&- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)^2 \\
\geq\ & 2(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)}))(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})) \\
&- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2 \\
&\times \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2 \\
&- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)^2
\end{aligned}
$$

$$
\begin{aligned}
= &\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right) \\
&\left(2 - \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)\right. \\
&\left. * \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)\right) \\
&- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)
\end{aligned}
$$

$$\times\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)$$

$$=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right.$$

$$\left(2-\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right.$$

$$\left.\left.*\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)\right]$$

$$-\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)$$

$$\times\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)$$

$$=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right.$$

$$\times\left(2-\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\right.$$

$$\left.\left.*\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right)\right]$$

$$-\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)$$

$$\times\left(\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\nu_q(x_{\sigma(j)})\right)$$

$$=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right.$$

$$\times\left(2-\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\right.$$

$$
\left.\left.* \sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right)\right]
$$

$$
-\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right.
$$

$$
\left.\times\left(\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\nu_q(x_{\sigma(j)})\right)\right)
$$

$$
=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right.
$$

$$
\left(2-\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\right.
$$

$$
\left.* \sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right)
$$

$$
\left.-\nu_q(x_{\sigma(i)})\left(\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\nu_q(x_{\sigma(j)})\right)\right]
$$

$$
=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right.
$$

$$
\times\left(2-\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\right.
$$

$$
\left.* \sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right)
$$

$$
\left.-\left(\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\nu_q(x_{\sigma(j)})\nu_q(x_{\sigma(i)})\right)\right]
$$

$$
=\sum_{i=1}^{n}(\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\left[\sum_{j=1}^{n}(\mu(A_{\sigma(j)})-\mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)})\right.
$$

$$\times \left(2 - \sum_{j=1}^{n}(\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j-1)}))\nu_p(x_{\sigma(j)})\right.$$

$$\left.\left. * \sum_{j=1}^{n}(\mu(A_{\sigma(j)}) - \mu(A_{\sigma(j-1)}))\nu_q(x_{\sigma(j)}) - \nu_p(x_{\sigma(j)})\nu_q(x_{\sigma(i)})\right)\right] \geq 0.$$

Consequently,

$$\left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2$$

$$+ \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2$$

$$- \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\right)^2$$

$$\times \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_q(x_{\sigma(i)})\right)^2$$

$$\geq \left(\sum_{i=1}^{n}(\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i-1)}))\nu_p(x_{\sigma(i)})\nu_q(x_{\sigma(i)})\right)^2.$$

According to Equation 4, this proof is completed. □

THEOREM 12. *Let $p(x_i) = (\mu_p(x_i), \nu_p(x_i))(i = 1, 2, \ldots, n)$ be a collection of PFNs on X, and μ be a fuzzy measure on X, then*

(1) $\mathrm{PFCIA}(p(x_1)^c, p(x_2)^c, \ldots, p(x_n)^c) = \mathrm{PFCIA}(p(x_1), p(x_2), \ldots, p(x_n))^c$*;*
(2) $\mathrm{PFCIG}(p(x_1)^c, p(x_2)^c, \ldots, p(x_n)^c) = \mathrm{PFCIG}(p(x_1), p(x_2), \ldots, p(x_n))^c$.

Proof. It is trial. □

4 PROPOSED PFCIA-BASED MABAC FOR MAGDM PROBLEMS

4.1. Description of the Problems

4.1.1. Attribute Is Dependent

Consider a MAGDM problem that contains a discrete set of m alternatives, expressed as $A = \{A_1, A_2, \ldots, A_m\}$. Suppose $D = \{d_1, d_2, \ldots, d_l\}$ be a set of l decision makers (DMs) that have the important degree of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, and $\lambda_k(k = 1, 2, \ldots, l)$ is a fuzzy number with $\sum_{k=1}^{n} \lambda_k = 1$. Let $C = \{c_1, c_2, \ldots, c_n\}$

be a collection of n attributes. Assume that $P^{(k)} = (p_{ij}^k)_{m\times n}$ be a Pythagorean fuzzy matrix, where p_{ij}^k is the PFN that the alternative $A_i \in A$ can take with respect to attribute $c_j \in C$ according to DM $d_k \in D$.

Various steps used in the proposed PFCIA operator for MAGDM are explained as follows:

Algorithm. (Attribute is dependent)

Step 1. Form a committee of the DMs, select the proper attributes, and obtain the prospective alternatives for the decision-making problem with the Pythagorean fuzzy decision matrix $P^{(k)} = (p_{ij}^k)_{m\times n} (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, l)$, $p_{ij}^k = (\mu_{ij}^k, \nu_{ij}^k)$.

Step 2. Construct the group Pythagorean fuzzy decision matrix $P = (p_{ij})_{m\times n} = (\mu_{ij}, \nu_{ij})_{m\times n}$.

$$p_{ij} = \mathrm{PFWA}(p_{ij}^1, p_{ij}^2, \ldots, p_{ij}^l) = \left(\sum_{k=1}^{l} \lambda_k \mu_{ij}^k, \sum_{k=1}^{l} \lambda_k \nu_{ij}^k\right). \tag{17}$$

Step 3. Normalize the Pythagorean fuzzy decision matrix $P = (p_{ij})_{m\times n}$ into $\widetilde{P} = (\widetilde{p}_{ij})_{m\times n} = (\widetilde{\mu}_{ij}, \widetilde{\nu}_{ij})_{m\times n}$.

$$\widetilde{p}_{ij} = \begin{cases} (\mu_{ij}, \nu_{ij}), & c_j \text{ is benefit attribute,} \\ (\nu_{ij}, \mu_{ij}), & c_j \text{ is cost attribute.} \end{cases} \tag{18}$$

Step 4. Reorder the $\widetilde{p}_{ij} (j = 1, 2, \ldots, n)$ for each alternative $A_i (i = 1, 2, \ldots, m)$ in a descending order by Equation 4 or 5.

Step 5. Confirm the fuzzy measures of attribute sets of C. We take Equation 6 for determining the fuzzy measure.

Step 6. Aggregate the Pythagorean fuzzy information $\widetilde{p}_i$ of alternative $A_i (i = 1, 2, \ldots, m)$.

$$\begin{aligned} \widetilde{p}_i &= \mathrm{PFICA}(\widetilde{p}_{i1}, \widetilde{p}_{i2}, \ldots, \widetilde{p}_{in}) \\ &= \left(\sum_{j=1}^{n} (\mu(C_{\sigma(j)}) - \mu(C_{\sigma(j-1)}))\widetilde{\mu}_{i\sigma(j)}, \sum_{j=1}^{n} (\mu(C_{\sigma(j)}) - \mu(C_{\sigma(j-1)}))\widetilde{\nu}_{i\sigma(j)}\right) \end{aligned} \tag{19}$$

or

$$\begin{aligned} \widetilde{p}_i &= \mathrm{PFICG}(\widetilde{p}_{i1}, \widetilde{p}_{i2}, \ldots, \widetilde{p}_{in}) \\ &= \left(\prod_{j=1}^{n} (\widetilde{\mu}_{i\sigma(j)})^{\mu(C_{\sigma(j)}) - \mu(C_{\sigma(j-1)})}, \prod_{j=1}^{n} (\widetilde{\nu}_{i\sigma(j)})^{\mu(C_{\sigma(j)}) - \mu(C_{\sigma(j-1)})}\right). \end{aligned} \tag{20}$$

Step 7. Compute the score function $s(\widetilde{p}_i)$ of $\widetilde{p}_i$ by Equation 4.

Step 8. Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$, the most preferred alternative is the one with the highest value of score function.

4.1.2. Attribute Is Independent

Consider a MAGDM problem that contains a discrete set of m alternatives, expressed as $A = \{A_1, A_2, \ldots, A_m\}$. Suppose $D = \{d_1, d_2, \ldots, d_l\}$ be a set of l DMs that have the important degree of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, and $\lambda_k (k = 1, 2, \ldots, l)$ is a fuzzy number with $\sum_{k=1}^{n} \lambda_k = 1$. Let $C = \{c_1, c_2, \ldots, c_n\}$ be a collection of n attributes with weight vector $w = (w_1, w_2, \ldots, w_n)$, and $\sum_{j=1}^{n} w_j = 1$. Assume that $P^{(k)} = (p_{ij}^k)_{m\times n}$ be a Pythagorean fuzzy matrix, where p_{ij}^k is the possible value that the alternative $A_i \in A$ can take with respect to attribute $c_j \in C$ according to DM $d_k \in D$.

Algorithm 2. (Attribute is independent)

Step 1. Form a committee of the DMs, select the proper attributes, and obtain the prospective alternatives for the decision-making problem with the Pythagorean fuzzy decision matrix $P^{(k)} = (p_{ij}^k)_{m\times n}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k = 1, 2, \ldots, l)$, $p_{ij}^k = (\mu_{ij}^k, \nu_{ij}^k)$.

Step 2. Construct the group Pythagorean fuzzy decision matrix $P = (p_{ij})_{m\times n} = (\mu_{ij}, \nu_{ij})_{m\times n}$.

$$p_{ij} = \mathrm{PFWA}\left(p_{ij}^1, p_{ij}^2, \ldots, p_{ij}^l\right) = \left(\sum_{k=1}^{l} \lambda_k \mu_{ij}^k, \sum_{k=1}^{l} \lambda_k \nu_{ij}^k\right). \tag{21}$$

Step 3. Normalize the Pythagorean fuzzy decision matrix $P = (p_{ij})_{m\times n}$ into $\widetilde{P} = (\widetilde{p}_{ij})_{m\times n} = (\widetilde{\mu}_{ij}, \widetilde{\nu}_{ij})_{m\times n}$.

$$\widetilde{p}_{ij} = \begin{cases} (\mu_{ij}, \nu_{ij}), & c_j \text{ is benefit attribute,} \\ (\nu_{ij}, \mu_{ij}), & c_j \text{ is cost attribute.} \end{cases} \tag{22}$$

Step 4. Calculate the weighted matrix $T = (t_{ij})_{m\times n}$ by Equation 23.

$$t_{ij} = (\mu'_{ij}, \nu'_{ij}) = w_j \widetilde{p}_{ij} = \left(\sqrt{1 - (1 - (\widetilde{\mu}_{ij})^2)^{w_j}}, (\widetilde{\nu}_{ij})^{w_j}\right). \tag{23}$$

Step 5. Determine the border approximation area matrix $G = (g_j)_{1\times n}$. The border approximation area (BAA) for each attribute is determined according to Equation 24.

$$g_j = \prod_{i=1}^{m} (t_{ij})^{1/m} = \left(\prod_{i=1}^{m} (\mu'_{ij})^{1/m}, \prod_{i=1}^{m} (\nu'_{ij})^{1/m}\right). \tag{24}$$

Step 6. Calculate the distance matrix $D = (d_{ij})_{m\times n}$ by Equation 25.

$$d_{ij} = \begin{cases} d(t_{ij}, g_j), & \text{if } t_{ij} > g_j, \\ 0, & \text{if } t_{ij} = g_j, \\ -d(t_{ij}, g_j), & \text{if } t_{ij} < g_j, \end{cases} \tag{25}$$

where distance measure d is defined as Equation 3.

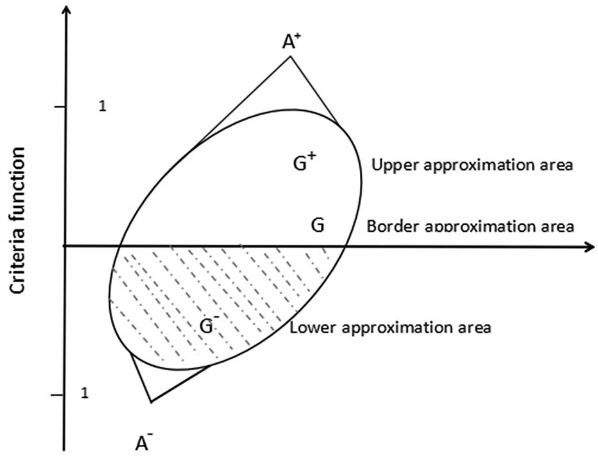

Figure 2. Presentation of the upper (G^+), lower (G^-), and border (G) approximation areas.

Especially, alternative A_i will belong to the border approximation area (G) if $d_{ij} = 0$, upper approximation area (G^+) if $d_{ij} > 0$, and lower approximation area (G^-) if $d_{ij} < 0$. The upper approximation area (G^+) is the area that contains the ideal alternative (A^+), whereas the lower approximation area (G^-) is the area that contains the anti-ideal alternative (A^-)(see Figure 2[46]). In order for alternative A_i to be selected as the best in the set, it is necessary for it to have as many attributes as possible belonging to the upper approximate area (G^+).

Step 7. Ranking the alternatives by $Q_i (i = 1, 2, \ldots, m)$. All the alternatives are ranked based on the descending order. The most preferred alternative is the one with the highest value of Q_i.

$$Q_i = \sum_{j=1}^{n} d_{ij}, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n, \tag{26}$$

where n is the number of attributes, m is the number of alternatives.

5. TWO NUMERICAL EXAMPLES

Example 1.[20] Listed Internet companies play an important role in China's stock market. The performance of listed companies affects resources allocation of capital market and has become a common concern of shareholders, creditors, government authorities, and other stakeholders. An investment bank wants to invest a sum of money in Internet stocks. So the investment bank arises three types of experts: market maker d_1, dealer d_2, finder d_3 to evaluate the potential investment value with the important degree of $\lambda = (0.2, 0.5, 0.3)$. They choose four Internet stocks in which the earnings ratio is higher than other stocks: (1) A_1 is SINA; (2) A_2 is BIDU; (3) A_3 is NETS; (4) A_4 is BABA from three attributes: (1) c_1 is the stock market trend; (2) c_2 is the policy direction; (3) c_3 is the annual performance. The attributes

Table I. Pythagorean fuzzy decision matrices $P^{(k)}(k = 1, 2, 3)$.

e_1	c_1	c_2	c_3
A_1	(0.9,0.3)	(0.8,0.1)	(0.9,0.2)
A_2	(0.5,0.7)	(0.4,0.7)	(0.8,0.1)
A_3	(0.3,0.5)	(0.8,0.4)	(0.3,0.8)
A_4	(0.6,0.7)	(0.5,0.6)	(0.4,0.2)
e_2	c_1	c_2	c_3
A_1	(0.7,0.2)	(0.9,0.2)	(0.7,0.2)
A_2	(0.6,0.7)	(0.5,0.6)	(0.6,0.2)
A_3	(0.7,0.1)	(0.6,0.5)	(0.8,0.4)
A_4	(0.6,0.6)	(0.5,0.8)	(0.6,0.4)
e_3	c_1	c_2	c_3
A_1	(0.8,0.1)	(0.9,0.2)	(0.8,0.1)
A_2	(0.7,0.6)	(0.6,0.4)	(0.7,0.2)
A_3	(0.9,0.4)	(0.8,0.6)	(0.7,0.4)
A_4	(0.8,0.6)	(0.7,0.5)	(0.6,0.4)

Table II. The group Pythagorean fuzzy decision matrix P.

e_1	c_1	c_2	c_3
A_1	(0.77,0.19)	(0.88,0.18)	(0.77,0.17)
A_2	(0.61,0.67)	(0.51,0.56)	(0.67,0.18)
A_3	(0.68,0.27)	(0.70,0.51)	(0.67,0.48)
A_4	(0.66,0.62)	(0.56,0.67)	(0.56,0.36)

are benefit attributes. The three experts $d_k(k = 1, 2, 3)$ evaluate the Internet stocks $A_i(i = 1, 2, 3, 4)$ with respect to the attributes $c_j(j = 1, 2, 3)$ and construct the following three Pythagorean fuzzy decision matrices $P^{(k)} = (p_{ij}^{(k)})_{4\times 3}$ as given in Table I.

Then, we utilize the Algorithm 1 to get the most desirable alternative(s), which involves the following steps:

Step 1. The prospective alternatives for the decision-making problem with the Pythagorean fuzzy decision matrix is shown in Table I.

Step 2. Compute the group Pythagorean fuzzy decision matrix $P = (p_{ij})_{4\times 3}$ by Equation 17, which is shown in Table II.

Step 3. Because the attributes are benefit attribute, there is no need to normalize.

Step 4. We rearrange the PFNs corresponding to each stock in a descending order by using Equation 4 or 5, which is shown as follows:

- $\widetilde{p}_{1\sigma(1)} = (0.88, 0.18)$, $\widetilde{p}_{1\sigma(2)} = (0.77, 0.17)$, $\widetilde{p}_{1\sigma(3)} = (0.77, 0.19)$,
- $\widetilde{p}_{2\sigma(1)} = (0.67, 0.18)$, $\widetilde{p}_{2\sigma(2)} = (0.51, 0.56)$, $\widetilde{p}_{2\sigma(3)} = (0.61, 0.67)$,
- $\widetilde{p}_{3\sigma(1)} = (0.68, 0.27)$, $\widetilde{p}_{3\sigma(2)} = (0.70, 0.51)$, $\widetilde{p}_{3\sigma(3)} = (0.67, 0.48)$,
- $\widetilde{p}_{4\sigma(1)} = (0.56, 0.36)$, $\widetilde{p}_{4\sigma(2)} = (0.66, 0.62)$, $\widetilde{p}_{4\sigma(3)} = (0.56, 0.67)$.

Step 5. Suppose following are the fuzzy measures of attribute of C.

$$\mu(c_1) = 0.5, \quad \mu(c_2) = 0.2, \quad \mu(c_3) = 0.3.$$

Parameter $\lambda = 0.5$ is obtained using Equation 6, and the following are obtained.

$$\mu(c_1, c_2) = 0.75, \quad \mu(c_1, c_3) = 0.875, \quad \mu(c_2, c_3) = 0.53, \quad \mu(c_1, c_2, c_3) = 1.$$

Step 6. Aggregate the Pythagorean fuzzy information $\widetilde{p}_i$ of alternative $A_i (i = 1, 2, 3, 4)$ by Equation 19, which is shown as follows:

$$\begin{aligned}\widetilde{p}_1 = \text{PFCIA}(\widetilde{p}_{11}, \widetilde{p}_{12}, \widetilde{p}_{13}) &= ((0.2 - 0) * 0.88 + (0.53 - 0.2) * 0.77 \\ &+ (1 - 0.53) * 0.77, (0.2 - 0) * 0.18 + (0.53 - 0.2) * 0.17 \\ &+ (1 - 0.53) * 0.19) = (0.7920, 0.1814),\end{aligned}$$

$$\begin{aligned}\widetilde{p}_2 = \text{PFCIA}(\widetilde{p}_{21}, \widetilde{p}_{22}, \widetilde{p}_{23}) &= ((0.3 - 0) * 0.67 + (0.53 - 0.3) * 0.51 \\ &+ (1 - 0.53) * 0.61, (0.3 - 0) * 0.18 + (0.53 - 0.3) * 0.56 \\ &+ (1 - 0.53) * 0.67) = (0.6050, 0.4977),\end{aligned}$$

$$\begin{aligned}\widetilde{p}_3 = \text{PFCIA}(\widetilde{p}_{31}, \widetilde{p}_{32}, \widetilde{p}_{33}) &= ((0.5 - 0) * 0.68 + (0.75 - 0.5) * 0.70 \\ &+ (1 - 0.75) * 0.67, (0.5 - 0) * 0.27 \\ &+ (0.75 - 0.5) * 0.51 + (1 - 0.75) * 0.48) = (0.6825, 0.3825),\end{aligned}$$

$$\begin{aligned}\widetilde{p}_4 = \text{PFCIA}(\widetilde{p}_{41}, \widetilde{p}_{42}, \widetilde{p}_{43}) &= ((0.3 - 0) * 0.56 + (0.875 - 0.3) * 0.66 \\ &+ (1 - 0.875) * 0.56, (0.3 - 0) * 0.36 + (0.875 - 0.3) * 0.62 \\ &+ (1 - 0.875) * 0.67) = (0.6175, 0.5483).\end{aligned}$$

Step 7. Compute the score function $s(\widetilde{p}_i)$ of $\widetilde{p}_i$ by Equation 4, which is shown as follows:

$$s(\widetilde{p}_1) = 0.5944, \quad s(\widetilde{p}_2) = 0.1183, \quad s(\widetilde{p}_3) = 0.3195, \quad s(\widetilde{p}_4) = 0.0807.$$

Step 8. Rank all the alternatives $A_i (i = 1, 2, 3, 4)$, we can obtain: $A_1 > A_3 > A_2 > A_4$. So the most preferred stock is A_1(SINA).

If we use the PFCIG operator, we can also get the most preferred stock is A_1(SINA), i.e., it is effective and feasible. Comparing with the existing approaches, our approach has more capability than [38,39] to model the vagueness in the practical MAGDM.

Example 2.[49] A venture capital company desires to invest in a R&D project. After the market research and preliminary screening have been conducted, there are three potential R&D projects $\{A_1, A_2, A_3\}$ for further evaluation. To reduce risk and increase profits, the company invites three DMs $D = \{d_1, d_2, d_3\}$ with the important degree of $\lambda = (0.2, 0.5, 0.3)$ to evaluate these three projects based on five attributes, including organizing ability (c_1), credit quality (c_2), level of research and development (c_3), profitability (c_4), and debt-servicing ability (c_5) with the attribute weight vector $w = (0.2, 0.1, 0.3, 0.15, 0.25)$. The three experts $d_k (k = 1, 2, 3)$ evaluate the R&D projects $A_i (i = 1, 2, 3)$ with respect to the attributes $c_j (j = 1, 2, 3, 4, 5)$ and construct the following three Pythagorean fuzzy decision matrices $P^{(k)} = (p_{ij}^{(k)})_{3\times 5}$ as listed in Table III.

Then, we utilize Algorithm 2 to get the most desirable alternative(s), which involves the following steps:

Table III. Pythagorean fuzzy decision matrices $P^{(k)}(k = 1, 2, 3)$.

e_1	c_1	c_2	c_3	c_4	c_5
A_1	(0.9,0.4)	(0.9,0.1)	(0.9,0.2)	(0.8,0.1)	(0.9,0.2)
A_2	(0.4,0.7)	(0.5,0.7)	(0.8,0.1)	(0.4,0.7)	(0.8,0.4)
A_3	(0.3,0.5)	(0.7,0.4)	(0.3,0.8)	(0.8,0.5)	(0.3,0.8)
e_2	c_1	c_2	c_3	c_4	c_5
A_1	(0.7,0.2)	(0.7,0.2)	(0.7,0.2)	(0.7,0.1)	(0.8,0.2)
A_2	(0.6,0.7)	(0.5,0.4)	(0.6,0.3)	(0.5,0.6)	(0.8,0.2)
A_3	(0.7,0.3)	(0.6,0.5)	(0.8,0.4)	(0.8,0.4)	(0.6,0.8)
e_3	c_1	c_2	c_3	c_4	c_5
A_1	(0.8,0.1)	(0.6,0.2)	(0.8,0.1)	(0.8,0.1)	(0.6,0.2)
A_2	(0.7,0.5)	(0.6,0.4)	(0.7,0.4)	(0.4,0.7)	(0.8,0.2)
A_3	(0.9,0.3)	(0.8,0.6)	(0.7,0.5)	(0.8,0.5)	(0.3,0.7)

Table IV. The group Pythagorean fuzzy decision matrix P.

e_1	c_1	c_2	c_3	c_4	c_5
A_1	(0.77,0.21)	(0.71,0.18)	(0.77,0.17)	(0.75,0.10)	(0.76,0.20)
A_2	(0.59,0.64)	(0.53,0.46)	(0.67,0.29)	(0.45,0.65)	(0.80,0.24)
A_3	(0.68,0.34)	(0.68,0.51)	(0.67,0.51)	(0.80,0.45)	(0.45,0.77)

Table V. The Pythagorean fuzzy decision matrix T.

e_1	c_1	c_2	c_3	c_4	c_5
A_1	(0.4056,0.7319)	(0.2603,0.8424)	(0.4861,0.5877)	(0.3415,0.7079)	(0.4403,0.6687)
A_2	(0.2864,0.9146)	(0.1801,0.9253)	(0.4046,0.6898)	(0.1827,0.9374)	(0.4748,0.6999)
A_3	(0.3417,0.8059)	(0.2453,0.9349)	(0.4046,0.8171)	(0.3769,0.8871)	(0.2345,0.9367)

Step 1. The prospective alternatives for the decision-making problem with the Pythagorean fuzzy decision matrix is shown in Table III.
Step 2. Compute the group Pythagorean fuzzy decision matrix $P = (p_{ij})_{3\times5}$ by Equation 21 as shown in Table IV.
Step 3. Because the attributes all are benefit attributes, there is no need to normalize.
Step 4. Calculate the elements from the weighted matrix $T = (t_{ij})_{3\times5}$ by Equation 23 as shown in Table V.
Step 5. Determine the border approximation area matrix $G = (g_j)_{1\times5}$. The BAA for each criterion is determined according to Equation 24 as shown below:

$$G = ((0.1518, 0.6514), (0.0692, 0.7924), (0.2288, 0.4564),$$
$$(0.1055, 0.6586), (0.1872, 0.5474)).$$

Step 6. Calculate the distance matrix $D = (d_{ij})_{3\times5}$ by Equation 25 as shown in Table VI
Step 7. Compute the $Q_i (i = 1, 2, 3)$ by Equation 26, we can obtain

$Q_1 = 0.9083 > Q_2 = -1.1927 > Q_3 = -2.2731$, i.e., A_1 is most preferred potential R&D project.

Table VI. The Pythagorean fuzzy decision matrix D.

e_1	c_1	c_2	c_3	c_4	c_5
A_1	0.2528	−0.1447	0.3210	0.1729	0.3063
A_2	−0.4712	−0.2559	−0.3789	−0.4673	0.3806
A_3	−0.3189	−0.3015	−0.5707	−0.4842	−0.5978

If we take the revised TOPSIS method, proposed by Zhang and Xu,[17] to instead of Steps 4–7, we can also obtain the A_1 is the most preferred potential R&D project, i.e., it is effective and feasible in our approach. Meanwhile, if we continue to aggregate the information of each alternative in different attributes by the PFWA operator after Step 3 and obtain their score function of each alternative, the preferred result is also A_1. Furthermore, it illustrates that our algorithm is practical.

6. CONCLUSIONS

Although many techniques have been introduced to aggregate Pythagorean fuzzy information,[19,31] all these existing Pythagorean fuzzy aggregation techniques only consider situations where all the elements in a PFS are independent, and thus these methods cannot be used to deal with many practical situations where the data under consideration are correlative; therefore, it is necessary to develop some new techniques to handle this issue. In this paper, we have used the Choquet integral to propose some operators for aggregating PFNs with correlative weights. The prominent characteristic of the operators is that they can not only consider the importance of the elements or their ordered positions but also reflect the correlations among the elements or their ordered positions. Most of the existing Pythagorean fuzzy aggregation operators, such as the PFWA operator,[19] PFWG operator, PFOWA operator, are special cases of our operators. A series of special cases of these two operators (PFCIA and PFCIG) have been discussed, and corresponding properties are explored in detail. Later, two approaches to MAGDM with attributes involving dependent and independent by the PFCIA operator and MABAC are proposed. Finally, we verify their effectiveness and practicality with two practical MAGDM problem.

Future studies will extend the proposed method to the MAGDM problems with hesitant fuzzy sets[50] or interval-valued Pythagorean fuzzy set.[31]

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References

1. Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst 1986;20:87–96.
2. Zadeh LA. Fuzzy sets. Inform control 1965;8:338–356.

3. Xu ZS. Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 2007;15:1179–1187.
4. Xu ZS, Yager RR. Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 2006;35:417–433.
5. Wei GW, Zhao XF. Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making. Expert Syst Appl 2012;39:2026–2034.
6. Wei GW. Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 2010;10:423–431.
7. Xu ZS, Yager RR. Intuitionistic fuzzy Bonferroni means. IEEE Trans Syst Man Cybernet 2011;41:568–578.
8. Xu ZS, Yager RR. Dynamic intuitionistic fuzzy multi-attribute decision making. Int J Approx Reason 2008;48:246–262.
9. Li BQ, He W. Intuitionistic fuzzy PRI-AND and PRI-OR aggregation operators. Inform Fusion 2013;14:450–459.
10. Yu DJ. Intuitionistic fuzzy prioritized operators and their application in multi-criteria group decision making. Technol Econ Dev Econ 2013;19:1–21.
11. Liao HC, Xu ZS. Intuitionistic fuzzy hybrid weighted aggregation operators. Int J Intell Syst 2014;29:971–993.
12. Yu DJ. Intuitionistic fuzzy geometric Heronian mean aggregation operators. Appl Soft Comput 2013;13:1235–1246.
13. Xia MM, Xu ZS, Zhu B. Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. Knowl-Based Syst 2012;31:78–88.
14. Yager RR. Pythagorean fuzzy subsets. In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada; June 24–28, 2013. pp 57–61.
15. Yager RR, Abbasov AM. Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 2014; 28:436–452.
16. Hadi-Vencheh A, Mirjaberi M. Fuzzy inferior ratio method for multiple attribute decision making problems. Inform Sci 2014; 277:263–272.
17. Zhang XL, Xu ZS. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 2014;29:1061–1078.
18. Yang Y, Ding H, Chen ZS, Li YL. A note on extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 2015;31:68–72.
19. Yager RR. Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst 2014;22:958–965.
20. Peng XD, Yang Y. Some results for Pythagorean fuzzy sets. Int J Intell Syst 2015;30:1133–1160.
21. Molodtsov D. Soft set theory—first results. Comput Math Appl 1999;37:19–31.
22. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning, I. Inform Sci 1975; 8:199–249.
23. Peng XD, Yang Y, Song JP, Jiang Y. Pythagorean fuzzy soft set and its application. Comput Eng 2015;41:224–229.
24. Peng XD, Yang Y. Multiple attribute group decision making methods based on Pythagorean fuzzy linguistic set. Comput Eng Appl. DOI:10.3778/j.issn.1002-8331.1501-0022.
25. Gou XJ, Xu ZS, Ren PJ. The properties of continuous Pythagorean fuzzy information. Int J Intell Syst. DOI:10.1002/int.21788.
26. Zhang XL. Multicriteria Pythagorean fuzzy decision analysis: a hierarchical QUALIFLEX approach with the closeness index-based ranking methods. Inform Sci 2016;330:104–124.
27. Beliakov G, James S. Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs. In: Proc 2014 IEEE Int Conf on Fuzzy Systems (FUZZ-IEEE), Beijing, People's Republic of China; July 6-11, 2014. pp 298–305.
28. Reformat MZ, Yager RR. Suggesting recommendations using Pythagorean fuzzy sets illustrated using Netflix movie data. In: Proc Information Processing and Management of

Uncertainty in Knowledge-Based Systems, Montpellier, France; July 15–19, 2014. pp 546–556.

29. Bustince H, Barrenechea E, Pagola M, Fernandez J, Xu ZS, Bedregal B, Montero J, Hagras H, Herrera F, De BB. A historical account of types of fuzzy sets and their relationships. IEEE Trans Fuzzy Syst, DOI:10.1109/TFUZZ.2015.2451692.
30. Li M, Yu YQ, Xu YC. Knowledge checking service selection method in Pythagorean fuzzy environment. In: Int Conf on Logistics Engineering, Management and Computer Science, Shenyang, People's Republic of China; July 29–31 , 2015. pp 1438–1443.
31. Peng XD, Yang Y. Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. Int J Intell Syst. DOI:10.1002/int.21790.
32. Sugeno M. Theory of fuzzy integral and its application (Doctoral dissertation). Tokyo Institute of Technology, Tokyo, Japan, 1974.
33. Kojadinovic I. Modelling interaction phenomena using fuzzy measure: on the notions of interaction and independence. Fuzzy Set Syst 2002;135:317–340.
34. Grabisch M. Alternative representations of discrete fuzzy measure for decision making. Int J Uncertain Fuzz 1997;5:587–607.
35. Ishii K, Sugeno M. A modal human evaluation process using measure. Int J Man Mach Stud 1985;22:19–38.
36. Choquet G. Theory of capacities. Ann Inst Fourier 1954;5:131–295.
37. Denneberg D. Non-additive measure and integral. Boston, MA: Kluwer; 1994.
38. Xu ZS. Choquet integrals of weighted intuitionistic fuzzy information. Inform Sci 2010;180:726–736.
39. Tan CQ, Chen XH. Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. Expert Syst Appl 2010;37:149–157.
40. Wu JZ, Chen F, Nie CP, Zhang Q. Intuitionistic fuzzy-valued Choquet integral and its application in multicriteria decision making. Inform Sci 2013;222:509–527.
41. Tan CP. A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS. Expert Syst Appl 2011;38:3023–3033.
42. Yu D, Wu Y, Zhou W. Multi-criteria decision making based on Choquet integral under hesitant fuzzy environment. J Comput Inform Syst 2011;7:4506–4513.
43. Peng JJ, Zhou H, Chen XH, Wang JQ. A multi-criteria decision-making approach based on TODIM and Choquet integral within a multiset hesitant fuzzy environment. Appl Math Inform Sci 2015;9:2087–2097.
44. Ju Y, Yang S, Liu X. Some new dual hesitant fuzzy aggregation operators based on Choquet integral and their applications to multiple attribute decision making. J Intell Fuzzy Syst 2014;27:2857–2868.
45. Joshi D, Kumar S. Interval-valued intuitionistic hesitant fuzzy Choquet integral based TOPSIS method for multi-criteria group decision making. Eur J Oper Res 2016;248:183–191.
46. Pamucar D, Cirovic G. The selection of transport and handling resources in logistics centers using multi-attributive border approximation area comparison (MABAC). Expert Syst Appl 2015;42:3016–3028.
47. Grabisch M, Murofushi T, Sugeno M. Fuzzy measure and integrals. New York: Physica-Verlag, 2000.
48. Grabisch M. Fuzzy integral in multicriteria decision making. Fuzzy Sets Syst 1995;69:279–298.
49. Wan SP, Xu GL, Wang F, Dong JY. A new method for Atanassov's interval-valued intuitionistic fuzzy MAGDM with incomplete attribute weight information. Inform Sci 2015;316:329–347.
50. Torra V. Hesitant fuzzy sets. Int J Intell Syst 2010;25:529–539.