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Weak M-Armendariz rings

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Abstract: For a monoid M, this paper introduces the weak M-Armendariz rings which are a common generalization of the M-Armendariz rings and the weak Armendariz rings, and investigates their properties. Moreover, this paper proves that a ring R is weak M-Armendariz if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ over R is weak M-Armendariz; if I is a semicommutative ideal of ring R such that R/I is weak M-Armendariz, then R is weak M-Armendariz, where M is a strictly totally ordered monoid; if a ring R is semicommutative and M-Armendariz, then R is weak $M\times N$-Armendariz, where N is a strictly totally ordered monoid; a finitely generated Abelian group G is torsion-free if and only if there exists a ring R such that R is weak G-Armendariz.

Key words: semicommutative rings; M-Armendariz rings; weak Armendariz rings; weak M-Armendariz rings

Throughout this paper R denotes an associative ring with identity, $\mathrm{nil}(R)$ denotes the set of all nilpotent elements of R and M denotes a monoid with identity e. Rege and Chhawchharia[1] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x)=a_0+a_1x+\dots+a_mx^m$, $g(x)=b_0+b_1x+\dots+b_nx^n\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i and j. The name "Armendariz ring" was chosen because Armendariz[2] had noted that a reduced ring satisfies this condition. Some properties, examples and counterexamples of Armendariz rings were given in Refs. [1-5]. A monoid M is called a u. p. -monoid (unique product monoid) if for any two nonempty finite subsets $A, B\subseteq M$, there exists an element $g\in M$ uniquely presented in the form ab where $a\in A$ and $b\in B$. Liu[6] called a ring R M-Armendariz if whenever elements $\alpha=a_1g_1+\dots+a_mg_m$, $\beta=b_1h_1+\dots+b_nh_n\in R[M]$ satisfy $\alpha\beta=0$, then $a_ib_j=0$ for each i and j, which is a generalization of Armendariz rings. He showed that a finite generated Abelian group G is torsion-free if and only if there exists a ring R such that R is G-Armendariz. He also showed that if R is a reduced and M-Armendariz ring, then R is $M\times N$-Armendariz, where N is a u. p. -monoid. Liu and Zhao[7] called a ring R weak Armendariz if whenever polynomials $f(x)=a_0+a_1x+\dots+a_mx^m$, $g(x)=b_0+b_1x+\dots+b_nx^n\in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j\in\mathrm{nil}(R)$ for each i and j. They showed that for a semicommutative ideal I such that R/I is weak Armendariz, then R is weak Armendariz, and R is weak Armendariz if and only if for any n, the $n\times n$ upper triangular matrix ring over R is weak Armendariz.

In this paper, a ring R is said to be weak M-Armendariz if whenever elements $\alpha=a_1g_1+\dots+a_mg_m$, $\beta=b_1h_1+\dots+b_nh_n\in R[M]$ satisfy $\alpha\beta=0$, then $a_ib_j\in\mathrm{nil}(R)$ for each i and j. Clearly, M-Armendariz rings are weak M-Armendariz. Examples are given to show that the converse is not always true. If $M=(\mathbb{N}\cup\{0\},+)$, weak Armendariz rings are weak M-Armendariz. If $M=\{e\}$, then every ring is M-Armendariz, so it is weak M-Armendariz. Thus weak M-Armendariz rings need not be weak Armendariz. Hence weak M-Armendariz rings are a common generalization of M-Armendariz rings and weak Armendariz rings. If S is a semigroup with multiplication $st=0$ for all $s, t\in S$ (for example, $S=\begin{bmatrix}0 & \mathbb{Z}\\ 0 & 0\end{bmatrix}$), and $M=S$, then any ring is not weak M-Armendariz. We show that R is weak M-Armendariz if and only if for any n, the $n\times n$ upper triangular matrix ring over R is weak M-Armendariz. It is shown that a finitely generated Abelian group G is torsion-free if and only if there exists a ring R with $|R|\geqslant 2$ such that R is weak G-Armendariz. This result weakens the second condition of theorem 1.14 in Ref. [6]. An ordered monoid $(M,\leqslant)$ is called a strictly ordered monoid if for any $g, g', h\in M$, $g<g'$ implies $gh<g'h$ and $hg<hg'$. For a strictly totally ordered monoid M, it is proved that if an ideal I is semicommutative such that R/I is weak M-Armendariz, then R is weak M-Armendariz. Moreover, for a monoid M and a strictly totally ordered monoid N, if R is a semicommutative and M-Armendariz ring, then R is weak $M\times N$-Armendariz.

1 Weak M-Armendariz Rings

Let $T_n(R)$ be the $n\times n$ upper triangular matrix over a ring R. In Ref. [7], Liu and Zhao showed that a ring R is weak Armendariz if and only if $T_n(R)$ is weak Armendariz for any n. If $M=(\mathbb{N}\cup\{0\},+)$, then R is weak M-Armendariz if and only if R is weak Armendariz. Moreover, note that every M-Armendariz ring is weak M-Armendariz. In the following we will give more examples of weak M-Armendariz rings which are not M-Armendariz.

Proposition 1 Let R be a ring and M a monoid. Then R is weak M-Armendariz if and only if, for any n, $T_n(R)$ is weak M-Armendariz.

Proof We note that any subring of weak M-Armendariz rings is weak M-Armendariz. Thus if $T_n(R)$ is a weak M-Armendariz ring, then R is a weak M-Armendariz ring.

Conversely, let $\alpha=A_1g_1+A_2g_2+\dots+A_pg_p$ and $\beta=B_1h_1+B_2h_2+\dots+B_qh_q$ be elements of $T_n(R)[M]$. Assume that $\alpha\beta=0$. It is easy to see that there exists an isomorphism of rings $T_n(R)[M]\to T_n(R[M])$ defined by

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$$\sum_{i=1}^{p}\begin{bmatrix} a_{11}^{i} & a_{12}^{i} & a_{13}^{i} & \cdots & a_{1n}^{i} \\ 0 & a_{22}^{i} & a_{23}^{i} & \cdots & a_{2n}^{i} \\ 0 & 0 & a_{33}^{i} & \cdots & a_{3n}^{i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{i} \end{bmatrix} g_i \mapsto$$

$$\begin{bmatrix} \sum_{i=1}^{p} a_{11}^{i} g_i & \sum_{i=1}^{p} a_{12}^{i} g_i & \sum_{i=1}^{p} a_{13}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{1n}^{i} g_i \\ 0 & \sum_{i=1}^{p} a_{22}^{i} g_i & \sum_{i=1}^{p} a_{23}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{2n}^{i} g_i \\ 0 & 0 & \sum_{i=1}^{p} a_{33}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{3n}^{i} g_i \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^{p} a_{nn}^{i} g_i \end{bmatrix}$$

Assume that

$$A_i=\begin{bmatrix} a_{11}^{i} & a_{12}^{i} & a_{13}^{i} & \cdots & a_{1n}^{i} \\ 0 & a_{22}^{i} & a_{23}^{i} & \cdots & a_{2n}^{i} \\ 0 & 0 & a_{33}^{i} & \cdots & a_{3n}^{i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{i} \end{bmatrix}$$

and

$$B_j=\begin{bmatrix} b_{11}^{j} & b_{12}^{j} & b_{13}^{j} & \cdots & b_{1n}^{j} \\ 0 & b_{22}^{j} & b_{23}^{j} & \cdots & b_{2n}^{j} \\ 0 & 0 & b_{33}^{j} & \cdots & b_{3n}^{j} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn}^{j} \end{bmatrix}$$

Then we have

$$\begin{bmatrix} \sum_{i=1}^{p} a_{11}^{i} g_i & \sum_{i=1}^{p} a_{12}^{i} g_i & \sum_{i=1}^{p} a_{13}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{1n}^{i} g_i \\ 0 & \sum_{i=1}^{p} a_{22}^{i} g_i & \sum_{i=1}^{p} a_{23}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{2n}^{i} g_i \\ 0 & 0 & \sum_{i=1}^{p} a_{33}^{i} g_i & \cdots & \sum_{i=1}^{p} a_{3n}^{i} g_i \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^{p} a_{nn}^{i} g_i \end{bmatrix}\circ$$

$$\begin{bmatrix} \sum_{j=1}^{p} b_{11}^{j} h_j & \sum_{j=1}^{p} b_{12}^{j} h_j & \sum_{j=1}^{p} b_{13}^{j} h_j & \cdots & \sum_{j=1}^{p} b_{1n}^{j} h_j \\ 0 & \sum_{j=1}^{p} b_{22}^{j} h_j & \sum_{j=1}^{p} b_{23}^{j} h_j & \cdots & \sum_{j=1}^{p} b_{2n}^{j} h_j \\ 0 & 0 & \sum_{j=1}^{p} b_{33}^{j} h_j & \cdots & \sum_{j=1}^{p} b_{3n}^{j} h_j \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{j=1}^{p} b_{nn}^{j} h_j \end{bmatrix}=0$$

It follows that

$$\left(\sum_{i=0}^{p} a_{ss}^{i} g_i\right)\left(\sum_{j=0}^{p} b_{ss}^{j} h_j\right)=0 \qquad s=1,2,\ldots,n$$

Since R is weak M-Armendariz, there exists $m_{ijs}\in N$ such that $(a_{ss}^{i}b_{ss}^{j})^{m_{ijs}}=0$ for any s, i and j. Let $m_{ij}=\max\{m_{ij1}, m_{ij2}, \ldots, m_{ijn}\}$, then

$$(A_iB_j)^{m_{ij}}=\begin{bmatrix} a_{11}^{i}b_{11}^{j} & * & * & \cdots & * \\ 0 & a_{22}^{i}b_{22}^{j} & * & \cdots & * \\ 0 & 0 & a_{33}^{i}b_{33}^{j} & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{i}b_{nn}^{j} \end{bmatrix}^{m_{ij}}=$$

$$\begin{bmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Thus $((A_iB_j)^{m_{ij}})^{n}=0$. This shows that $T_n(R)$ is a weak M-Armendariz ring.

Corollary 1 Let M be a monoid. If a ring R is an M-Armendariz ring, then, for any n, $T_n(R)$ is a weak M-Armendariz ring.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M)=R\oplus M$ with the usual addition and the multiplication: $(r_1, m_1)(r_2, m_2)=(r_1r_2, r_1m_2+m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$, where $r\in R$ and $m\in M$ and the usual matrix operations are used.

Proposition 2 Let M be a monoid. Then R is a weak M-Armendariz ring if and only if the trivial extension $T(R, R)$ is a weak M-Armendariz ring.

Proof It follows from proposition 1.

In general, for any ring R, the $n\times n$ $(n\geqslant 2)$ full matrix ring $M_n(R)$ over R need not be a weak M-Armendariz ring as shown by the following example.

Example 1 Let R be a ring and M a monoid with $|M|\geqslant 2$. Let $S=M_2(R)$. Take $e\neq g\in M$. Let $\alpha=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}e+\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}g$ and $\beta=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}e+\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}g$ in $S[M]$. Then we have $\alpha\beta=0$. But $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not nilpotent. Thus S is not weak M-Armendariz.

Proposition 3 Let M be a cancellative monoid and N be an ideal of M. If a ring R is weak N-Armendariz, then R is weak M-Armendariz.

Proof Let $\alpha=a_1g_1+\cdots+a_mg_m$, $\beta=b_1h_1+\cdots+b_nh_n$ in $R[M]$ with $\alpha\beta=0$. Set $g\in N$, then $gg_1, gg_2, \ldots, gg_m, h_1g, h_2g, \ldots, h_ng\in N$ and $gg_i\neq gg_j$ and $h_ig\neq h_jg$ when $i\neq j$. Now from $\left(\sum_{i=1}^{m} a_igg_i\right)\left(\sum_{j=1}^{n} b_jh_jg\right)=0$ and the hypothesis that R is weak N-Armendariz, it follows that $a_ib_j\in nil(R)$ for all i and j. Thus R is weak M-Armendariz.

Proposition 4 For a ring R and a monoid M, suppose that R/I is weak M-Armendariz for some ideal I of R. If $I\subseteq nil(R)$, then R is weak M-Armendariz.

Proof Let $\alpha=a_1g_1+a_2g_2+\cdots+a_mg_m$, $\beta=b_1h_1+b_2h_2+\cdots+b_nh_n\in R[M]$ such that $\alpha\beta=0$. Then $\left(\sum_{i=1}^{m}\bar{a}_ig_i\right)\left(\sum_{j=0}^{n}\bar{b}_jh_j\right)=0$. Thus $(\bar{a}_i\bar{b}_j)^{n}=0$ for some positive integer n_{ij}. Hence $a_ib_j\in nil(R)$. This means that R is weak M-Armendariz.

Proposition 5 For a monoid M, if R is a finite subdirect sum of weak M-Armendariz rings, then R is weak M-Armendariz.

Proof Let I_k $(k=1,2,\cdots,l)$ be ideals of R such that R/I_k is weak M-Armendariz and $\bigcap_{k=1}^{l} I_k=0$. Suppose that $\alpha=\sum_{i=0}^{m} a_ig_i$ and $\beta=\sum_{j=0}^{n} b_jh_j\in R[M]$ are such that $\alpha\beta=0$. Then there exists $n_{ijk}\in N$ such that $(\bar{a}_i\bar{b}_j)^{n_{ijk}}=0$ in R/I_k. Thus $(a_ib_j)^{n_{ijk}}\in I_k$. Set $n_{ij}=\max\{n_{ij1},n_{ij2},\cdots,n_{ijl}\}$, then $(a_ib_j)^{n_{ij}}\in I_k$ for any k, which implies that $(a_ib_j)^{n_{ij}}=0$. Thus R is weak M-Armendariz.

Recall that R is semicommutative if $ab=0$ implies $aRb=0$ for $a,b\in R$. An ideal I of R is semicommutative if it is semicommutative as a ring without identity. In Ref.[7], Liu and Zhao proved that if I is a semicommutative ideal of R such that R/I is weak Armendariz, then R is weak Armendariz. The following result is a generalization of this.

Theorem 1 For a ring R and a strictly totally ordered monoid M, suppose that R/I is weak M-Armendariz for some ideal I of R. If I is semicommutative, then R is weak M-Armendariz.

Proof Let $\alpha,\beta\in R[M]$ be such that $\alpha\beta=0$. We write $\alpha=a_1g_1+a_2g_2+\cdots+a_mg_m$, $\beta=b_1h_1+b_2h_2+\cdots+b_nh_n$ with $g_1<g_2<\cdots<g_m$ and $h_1<h_2<\cdots<h_n$. We will use transfinite induction on the strictly totally ordered set $(M,\leqslant)$ to show that $a_ib_j\in nil(R)$ for any i and j. Note that $(\bar{a}_1g_1+\bar{a}_2g_2+\cdots+\bar{a}_mg_m)(\bar{b}_1h_1+\bar{b}_2h_2+\cdots+\bar{b}_nh_n)=0$ in $(R/I)[M]$. Since R/I is weak M-Armendariz, there exists a positive integer n_{ij} such that $(a_ib_j)^{n_{ij}}\in I$. Clearly $g_1h_1<g_ih_j$ if $i\neq 1$ or $j\neq 1$. Hence $a_1b_1=0\in nil(R)$. Now suppose that $w\in M$ is such that for any g_i and h_j with $g_ih_j<w$, $a_ib_j\in nil(R)$. We will show that $a_ib_j\in nil(R)$ for any g_i and h_j with $g_ih_j=w$. Set $X=\{(g_i,h_j)\mid g_ih_j=w\}$. Then X is a finite set. We write X as $\{(g_{i_t},h_{j_t})\mid t=1,2,\cdots,k\}$ such that $g_{i_1}<g_{i_2}<\cdots<g_{i_k}$. Since M is cancellative, $g_{i_1}=g_{i_2}$ and $g_{i_1}h_{j_1}=g_{i_2}h_{j_2}=w$ imply $h_{j_1}=h_{j_2}$. Since $\leqslant$ is a strict order, $g_{i_1}<g_{i_2}$ and $g_{i_1}h_{j_1}=g_{i_2}h_{j_2}=w$ imply $h_{j_2}<h_{j_1}$. Thus we have $h_{j_k}<\cdots<h_{j_2}<h_{j_1}$. Now

$$\sum_{(g_i,h_j)\in X} a_ib_j=\sum_{t=1}^{k} a_{i_t}b_{j_t}=0$$

For any $t\geqslant 2$, $g_{i_1}h_{j_t}<g_{i_t}h_{j_t}=w$, and thus by induction hypothesis we have $a_{i_1}b_{j_t}\in nil(R)$. Let $p=n_{i_1j_1}$. Then $(a_{i_1}b_{j_1})^p\in I$. By hypothesis, $a_{i_1}b_{j_2}\in nil(R)$. Let $(a_{i_1}b_{j_2})^{p'}=0$. Then $(b_{j_2}a_{i_1})^{p'+1}=0$. Thus

$$(a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1}a_{i_2}(b_{j_2}a_{i_1})^{p'+1}(b_{j_2}(a_{i_1}b_{j_1})^{p+1})=0$$

Since $(a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1}a_{i_2}(b_{j_2}a_{i_1})\in I$, $(b_{j_2}a_{i_1})^{p'}(b_{j_2}\cdot(a_{i_1}b_{j_1})^{p+1})\in I$, $b_{j_1}(a_{i_1}b_{j_1})^pa_{i_2}\in I$ and I is semicommutative, it follows that

$$((a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1}a_{i_2}(b_{j_2}a_{i_1}))(b_{j_1}(a_{i_1}b_{j_1})^pa_{i_2})\cdot((b_{j_2}a_{i_1})^{p'}(b_{j_2}(a_{i_1}b_{j_1})^{p+1}))=0$$

That is

$$((a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1})((a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1})\cdot a_{i_2}(b_{j_2}a_{i_1})^{p'}(b_{j_2}(a_{i_1}b_{j_1})^{p+1})=0$$

$$((a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1})^2a_{i_2}(b_{j_2}a_{i_1})^{p'}(b_{j_2}(a_{i_1}b_{j_1})^{p+1})=0$$

Continuing this procedure, it yields that $((a_{i_2}b_{j_2})\cdot(a_{i_1}b_{j_1})^{p+1})^{p'+3}=0$. Thus $(a_{i_2}b_{j_2})(a_{i_1}b_{j_1})^{p+1}\in nil(I)$. Similarly we can show that $(a_{i_t}b_{j_t})(a_{i_1}b_{j_1})^{p+1}\in nil(I)$ for $t=3,\cdots,k$. By lemma 3.1 in Ref.[7], nil(I) is an ideal of I since I is semicommutative. Thus, if we multiply the equation $\sum_{t=1}^{k} a_{i_t}b_{j_t}=0$ on the right side by $(a_{i_1}b_{j_1})^{p+1}$, then

$$(a_{i_1}b_{j_1})^{p+2}=-\left(\sum_{t=2}^{k} a_{i_t}b_{j_t}(a_{i_1}b_{j_1})^{p+1}\right)\in nil(I)$$

Thus $a_{i_1}b_{j_1}\in nil(R)$. Let $q=n_{i_2j_2}$, then $(a_{i_2}b_{j_2})^q\in I$. By analogy with the above proof, we have $\sum_{t=3}^{k} a_{i_t}b_{j_t}(a_{i_2}b_{j_2})^{q+1}\in nil(I)$. Suppose that $(a_{i_1}b_{j_2})^s=0$. Then

$$(a_{i_2}b_{j_2})^{q+1}(a_{i_1}b_{j_1})^s(a_{i_2}b_{j_2})^{q+1}=0$$

Since $(a_{i_2}b_{j_2})^{q+1}\in I$ and I is semicommutative, we have

$$((a_{i_1}b_{j_1})(a_{i_2}b_{j_2})^{q+1})^{s+1}=0$$

Thus $(a_{i_1}b_{j_1})(a_{i_2}b_{j_2})^{q+1}\in nil(I)$. Hence, multiplying the equation $\sum_{t=1}^{k} a_{i_t}b_{j_t}=0$ on the right side by $(a_{i_2}b_{j_2})^{q+1}$, we have

$$(a_{i_2}b_{j_2})^{q+2}=-\left(\sum_{t=3}^{k} a_{i_t}b_{j_t}(a_{i_2}b_{j_2})^{q+1}\right)-(a_{i_1}b_{j_1})(a_{i_2}b_{j_2})^{q+1}\in nil(I)$$

Hence, $a_{i_2}b_{j_2}\in nil(R)$. Similarly we can show that $a_{i_3}b_{j_3}\in nil(R),\cdots,a_{i_k}b_{j_k}\in nil(R)$. Thus $a_{i_t}b_{j_t}\in nil(R)$ for any t with $g_{i_t}h_{j_t}=w$. Therefore, by transfinite induction, $a_ib_j\in nil(R)$ for any i and j. Thus, R is weak M-Armendariz.

Corollary 2 Let M be a strictly totally ordered monoid and R a semicommutative ring. Then R is weak M-Armendariz.

By corollary 3.4 in Ref.[7], semicommutative rings are weak Armendariz. In the following, we will see that semicommutative rings need not be weak M-Armendariz. Thus the condition that M is a strictly totally ordered monoid in theorem 1 is not superfluous.

Proposition 6 If M is a finite monoid, then the complex field C is not weak M-Armendariz.

Proof Suppose that C is weak M-Armendariz. Let $M=\{e,g_1,\cdots,g_n\}$. Let $\alpha=a_0e+a_1g_1+\cdots+a_ng_n$ and $\beta=b_0e+b_1g_1+\cdots+b_ng_n\in C[M]$ such that $\alpha\beta=0$. Then $a_ib_j\in nil(C)$. Note that C is reduced, so $a_ib_j=0$. Thus, C is M-Armendariz, which contradicts proposition 1.15 in Ref.[6].

Corollary 3 Let R be a semicommutative ring. Then R is weak Z-Armendariz.

Recall that a monoid M is torsion-free if for any $g,h\in M$ and $k\geqslant 1$, $g^k=h^k$ implies $g=h$.

Corollary 4 Let M be a commutative, cancellable and torsion-free monoid. If one of the following conditions holds, then R is a weak M-Armendariz ring:

1) R is semicommutative;

2) R/I is weak M-Armendariz for some ideal I of R and I is semicommutative.

Proof If M is commutative, cancellative and torsion-free, then there exists a compatible strict total order $\leqslant$ on M[8]. Now the results follow from theorem 1.

Lemma 1 Let M be a monoid and N be a submonoid of M. If R is a weak M-Armendariz ring, then R is weak N-Armendariz.

Lemma 2 Let M be a cyclic group of order $n\geqslant 2$ and R be a ring with $0\neq 1$. Then R is not weak M-Armendariz.

Proof Suppose that $M=\{e, g, g^2, \dots, g^{n-1}\}$. Let $\alpha = 1e + 1g + 1g^2 + \dots + 1g^{n-1}$ and $\beta = 1e + (-1)g$, then $\alpha\beta = 0$, but $1\cdot 1 = 1$ is not a nilpotent element. Thus R is not weak M-Armendariz.

Let $T(G)$ be the set of elements of the finite order in an Abelian group G. Then $T(G)$ is a fully invariant subgroup of G. G is torsion-free if and only if $T(G)=\{e\}$. Liu[6] showed that for a finitely generated Abelian group G, it is torsion-free if and only if there exists a ring R with $|R|\geqslant 2$ such that R is G-Armendariz. The following theorem will weaken the sufficient condition of this.

Theorem 2 Let G be a finitely generated Abelian group. Then the following conditions on G are equivalent:

1) G is torsion-free;

2) There exists a ring R with $|R|\geqslant 2$ such that R is weak G-Armendariz.

Proof 1)$\Rightarrow$2) If G is a finitely generated Abelian group with $T(G)=\{e\}$, then $G\cong Z\times Z\times \dots \times Z$, a finite direct product of group Z. It is easy to see that $Z\times Z\times \dots\times Z$ is a commutative, cancellative and torsion-free monoid, so is G. Let R be a semicommutative ring. Then, by corollary 4, R is weak G-Armendariz.

2)$\Rightarrow$1) If $g\in T(G)$ and $g\neq e$, then $N=\langle g\rangle$ is a cyclic group of the finite order. If a ring $R\neq\{0\}$ is weak G-Armendariz, then by lemma 1, R is weak N-Armendariz, contradicting lemma 2. Thus every ring $R\neq\{0\}$ is not weak G-Armendariz.

Let M be a monoid and N be an ideal of M. Denote the Rees congruence induced by N by $\rho(N)$, which is defined by $g\rho(N)h \Leftrightarrow g, h\in N$ or $g=h$. If R is weak M-Armendariz, then by lemma 1, R is weak N-Armendariz since $R[N]$ is a subring of $R[M]$. But R may not be weak $M/\rho(N)$-Armendariz as shown by the following example.

Example 2[6] Let $M=\{\mathbf{N}\cup\{0\}, +\}$ and $N=\{2, 3, \dots\}$. Then N is an ideal of M and $M/\rho(N)$ is a finite monoid. From proposition 6, it follows that C is not weak $M/\rho(N)$-Armendariz, but it is weak M-Armendariz because C is M-Armendariz.

2 Monoid Rings

Lemma 3 Let R be a semicommutative ring and M a monoid. If $a_1, \dots, a_n\in \mathrm{nil}(R)$, then $a_1g_1 + \dots + a_ng_n\in \mathrm{nil}(R[M])$.

Proof The proof is similar to that of lemma 3.7 in Ref.[7].

For a monoid M, we denote the largest subgroup of M by $G(M)$. In Ref.[7], Liu and Zhao proved that if R is semicommutative, then $R[x]$ is weak Armendariz. In Ref.[6], Liu proved that if M is a commutative and cancellative monoid with $G(M)=\{e\}$ and R is Armendariz and M-Armendariz, then $R[M]$ is Armendariz. For the above monoid M, we do not know whether $R[M]$ is weak Armendariz if R is weak Armendariz and weak M-Armendariz. However, if R is semicommutative, we have the following proposition.

Proposition 7 Let M be a commutative and cancellative monoid with $G(M)=\{e\}$. If R is a semicommutative and weak M-Armendariz ring, then $R[M]$ is weak Armendariz.

Proof Suppose that $\left(\sum_{i=0}^{m}\alpha_i x^i\right)\left(\sum_{j=0}^{n}\beta_j x^j\right)=0$, where $\alpha_i=\sum a_{ip}g_{ip}$, $\beta_j=\sum b_{jq}h_{jq}\in R[M]$. Set $g=\left(\prod_i\prod_p g_{ip}\right)\left(\prod_j\prod_q g_{jq}\right)$. Clearly for any $r\in R$ and $h\in M$, $(rh)(1g)=(1g)(rh)$. Thus from $\left(\sum_{i=0}^{m}\alpha_i x^i\right)\left(\sum_{j=0}^{n}\beta_j x^j\right)=0$, it follows that $\left(\sum_{i=0}^{m}\alpha_j(1g)^i\right)\left(\sum_{j=0}^{n}\beta_j(1g)^j\right)$. Thus we have

$$\left(\sum_i\sum_p a_{ip}g_{ip}g^i\right)\left(\sum_j\sum_q b_{jq}h_{jq}g^j\right)=0$$

Suppose that $g_{i'p'}g^{i'}=g_{i''p''}g^{i''}$ for some i' and i''. If $i'=i''$, then $g_{i'p'}=g_{i''p''}$ since M is cancellative, and so $p'=p''$. Thus without loss of generality, we can assume that $i'>i''$. Then $g_{i'p'}g^{(i'-i'')}=g_{i''p''}$ since M is cancellative. Thus it is easy to see that g_{ip} and h_{jq} are in $G(M)$ for all i, j, p, q. Hence $g_{ip}=h_{jq}=e$ by hypothesis, and then we may assume that $\alpha_i=a_ie$ and $\beta_j=b_je$ for all i, j. So we have

$$\left(\sum_i (a_ie)x^i\right)\left(\sum_j (b_je)x^j\right)=0$$

from which it follows that $\left(\sum_i a_ix^i\right)\left(\sum_j b_jx^j\right)=0$. Thus $a_ib_j\in\mathrm{nil}(R)$ for all i and j since R is weak Armendariz. Assume that $(a_ib_j)^{n_{ij}}=0$ for some $n_{ij}\in \mathbf{N}$. Then $(\alpha_i\beta_j)^{n_{ij}}=(a_ib_j)^{n_{ij}}e=0$. So $\alpha_i\beta_j\in\mathrm{nil}(R[M])$. If $h_{jq'}g^{j'}=h_{jq''}g^{j''}$ for some j' and j'', then by analogy with the above proof, it follows that $\alpha_i\beta_j=(a_ie)(b_je)\in\mathrm{nil}(R[M])$ for all i, j. Now suppose that each pair of $g_{ip}g^{i}$'s is distinct and each pair of $h_{jq}g^j$'s is distinct. Then $a_{ip}b_{jq}\in\mathrm{nil}(R)$ for all i, j, p, q since R is weak M-Armendariz. Thus $\alpha_i\beta_j=\sum_p\sum_q(a_{ip}b_{jq})(g_{ip}h_{jq})\in\mathrm{nil}(R[M])$ by lemma 3.

We can easily obtain the following fact.

Lemma 4 Let M be a monoid. If R is a semicommutative and M-Armendariz ring, then $R[M]$ is semicommutative.

Let $M=\{e\}$ and $N=\{\mathbf{N}\cup\{0\}, +\}$. Let R be a semicommutative ring. Then R is M-Armendariz. But $R[M]$ need not be N-Armendariz by example 3.5 in Ref.[1]. However, we have the following result.

Proposition 8 Let M be a monoid and N a strictly totally ordered monoid. If R is a semicommutative and M-Armendariz ring, then $R[M]$ is a weak N-Armendariz ring.

Proof Since R is semicommutative and M-Armendariz, by lemma 4, $R[M]$ is semicommutative. The assertion holds according to corollary 2.

Corollary 5 Let N be a strictly totally ordered monoid

If R is a semicommutative and Armendariz ring, then $R[x]$ is a weak N-Armendariz ring.

Proposition 9 Let M be a monoid and N a strictly totally ordered monoid. If R is a semicommutative and M-Armendariz ring, then $R[N]$ is a weak M-Armendariz ring.

Proof It is easy to see that there exists an isomorphism of rings $R[N][M]\to R[M][N]$ defined by $\sum_p\left(\sum_i a_{pi} n_i\right) m_p \mapsto \sum_i\left(\sum_p a_{ip} m_p\right) n_i$.

Now suppose that $\alpha_i, \beta_j \in R[N]$ are such that $\left(\sum_i \alpha_i m_i\right)\left(\sum_j \beta_j m'_j\right) = 0$. We will show that $\alpha_i\beta_j \in nil(R[N])$ for all i, j. Assume that $\alpha_i = \sum_p a_{ip} n_p$ and $\beta_j = \sum_q b_{jq} n'_q$ where $n_p, n'_q \in N$ for all p and q. Then $\left(\sum_i\left(\sum_p a_{ip} n_p\right) m_i\right)\left(\sum_j\left(\sum_q b_{jq} n'_q\right) m'_j\right) = 0$. Thus in $R[M][N]$ we have $\left(\sum_p\left(\sum_i a_{ip} m_i\right) n_p\right)\left(\sum_q\left(\sum_j b_{jq} m'_j\right) n'_q\right) = 0$. By proposition 8, $R[M]$ is weak N-Armendariz, $\left(\sum_i a_{ip} m_i\right)\left(\sum_j b_{jq} m'_j\right) \in nil(R[M])$ for all p, q. Since R is M-Armendariz, $a_{ip} b_{jq} \in nil(R)$ for all i, j, p, q according to proposition 1.6 in Ref.[6]. Hence $\alpha_i\beta_j \in nil(R[N])$ by lemma 3. This means that $R[N]$ is weak M-Armendariz.

Corollary 6 Let M be a monoid and R be a semicommutative ring. If R is M-Armendariz, then $R[x]$ and $R[x, x^{-1}]$ are weak M-Armendariz.

Proof Note that $R[x] \cong R[N\cup\{0\}]$ and $R[x, x^{-1}] \cong R[Z]$.

In Ref.[6], Liu showed that if R is reduced and M-Armendariz, then R is $M\times N$-Armendariz, where N is a u.p.-monoid. For weak M-Armendariz rings, we have the following result.

Theorem 3 Let M be a monoid and N be a strictly totally ordered monoid. If R is a semicommutative and M-Armendariz ring, then R is weak $M\times N$-Armendariz.

Proof Suppose that $\sum_{i=1}^{s} a_i(m_i, n_i)$ is in $R[M\times N]$. For any $1\leqslant p\leqslant t$, denote $A_p = \{i \mid 1\leqslant i\leqslant s, n_i = n_p\}$. Then $\sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p \in R[M][N]$. Note that $m_i \neq m_{i'}$ for any $i, i'\in A_p$ with $i\neq i'$. Now it is easy to see that there exists an isomorphism of rings $R[M\times N]\to R[M][N]$ defined by $\sum_{i=1}^{s} a_i(m_i, n_i)\mapsto \sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p$.

Suppose that $\left(\sum_{i=1}^{s} a_i(m_i, n_i)\right)\left(\sum_{j=1}^{s'} b_j(m'_j, n'_j)\right) = 0$ in $R[M\times N]$. Then from the above isomorphism, it follows that

$$\left(\sum_{p=1}^{t}\left(\sum_{i\in A_p} a_i m_i\right) n_p\right)\left(\sum_{q=1}^{t'}\left(\sum_{j\in B_q} b_j m'_j\right) n'_q\right) = 0$$

By proposition 8, $R[M]$ is weak N-Armendariz, thus we have $\left(\sum_{i\in A_p} a_i m_i\right)\left(\sum_{j\in B_q} b_j m'_j\right) \in nil(R[M])$ for all p, q. Since R is M-Armendariz, $a_i b_j \in nil(R)$ for any $i\in A_p$ and $j\in B_q$ by proposition 1.6 in Ref.[6]. Hence $a_i b_j \in nil(R)$ for all $1\leqslant i\leqslant s$ and $1\leqslant j\leqslant s'$. The proof is completed.

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摘要：对于幺半群 M，引入了弱 M-Armendariz 环的概念，此概念是 M-Armendariz 环和弱 Armendariz 环的共同推广．研究了这类环的性质，并且证明了：R 是弱 M-Armendariz 环当且仅当对任意的 n，R 的 n 阶上三角矩阵环 $T_n(R)$ 是弱 M-Armendariz 环；如果 I 是环 R 的半交换理想，使得 R/I 是弱 M-Armendariz 环，则 R 是弱 M-Armendariz 环，其中 M 是严格全序幺半群；如果 R 是半交换的 M-Armendariz 环，则 R 是弱 M×N-Armendariz 环，其中 N 是严格全序幺半群；有限生成 Abelian 群 G 是 torsion-free 的当且仅当存在一个环 R，使得 R 是弱 G-Armendariz 环．

关键词：半交换环；M-Armendariz 环；弱 Armendariz 环；弱 M-Armendariz 环

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